Y2P11 XMQs and MS

(Total: 229 marks)

1. P1_Sample Q4 . 4 marks - Y2P11 Integration
2. P1_Sample Q14. 10 marks - Y2P11 Integration
3. P2_Sample Q16. 12 marks - Y2P11 Integration
4. P1_Specimen Q1 . 6 marks - Y2P11 Integration
5. P1_Specimen Q7 . 5 marks - Y2P11 Integration
6. P1_Specimen Q12. 7 marks - Y2P11 Integration
7. P2_Specimen Q10. 9 marks - Y2P11 Integration
8. P1_2018 Q7 . 7 marks - Y2P11 Integration
9. P1_2018 Q10. 8 marks - Y2P11 Integration
10. P1_2018 Q13. 7 marks - Y2P11 Integration
11. P2_2018 Q10. 8 marks - Y2P11 Integration
12. P2_2018 Q13. 10 marks - Y2P11 Integration
13. P1_2019 Q13. 11 marks - Y2P1 Algebraic methods
14. P2_2019 Q2 . 4 marks - Y2P11 Integration
15. P2_2019 Q5 . 3 marks - Y2P11 Integration
16. P2_2019 Q14. 15 marks - Y2P11 Integration
17. P1_2020 Q10. 10 marks - Y2P11 Integration
18. P1_2020 Q14. 10 marks - Y2P11 Integration
19. P2_2020 Q1 . 5 marks - Y2P11 Integration
20. P2_2020 Q6 . 7 marks - Y2P1 Algebraic methods
21. P2_2020 Q12. 11 marks - Y2P11 Integration
22. P1_2021 Q11. 8 marks - Y2P11 Integration
23. P2_2021 Q12. 7 marks - Y2P11 Integration
24. P2_2021 Q14. 12 marks - Y2P11 Integration

25. P1_2022 Q4 . 3 marks - Y1P13 Integration

26. P1_2022 Q12. 5 marks - Y2P11 Integration

27. P1_2022 Q16. 9 marks - Y2P11 Integration

28. P2_2022 Q5 . 6 marks - Y2P11 Integration

29. P2_2022 Q14. 10 marks - Y2P11 Integration

4. Given that a is a positive constant and

$$\int_a^{2a} \frac{t+1}{t}\,\mathrm{d}t = \ln 7$$

show that $a = \ln k$, where k is a constant to be found.

(4)

(Total for Question 4 is 4 marks)

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Question	Scheme	Marks	AOs
3(a)	Attempts $(x-2)^2+(y+5)^2=....$	M1	1.1b
	Centre (2, −5)	A1	1.1b
		(2)	
(b)	Sets $k+2^2+5^2>0$	M1	2.2a
	$\Rightarrow k>-29$	A1ft	1.1b
		(2)	
			(4 marks)

Notes:

(a)

M1: Attempts to complete the square so allow $(x-2)^2+(y+5)^2=....$

A1: States the centre is at (2, −5). Also allow written separately $x=2, y=-5$

(2, −5) implies both marks

(b)

M1: Deduces that the right hand side of their $(x\pm...)^2+(y\pm...)^2=...$ is >0 or $\geqslant 0$

A1ft: $k>-29$ Also allow $k\geqslant -29$ Follow through on their rhs of $(x\pm...)^2+(y\pm...)^2=...$

Question	Scheme	Marks	AOs
4	Writes $\int \frac{t+1}{t}\,\mathrm{d}t=\int 1+\frac{1}{t}\,\mathrm{d}t$ and attempts to integrate	M1	2.1
	$=t+\ln t\ (+c)$	M1	1.1b
	$(2a+\ln 2a)-(a+\ln a)=\ln 7$	M1	1.1b
	$a=\ln\frac{7}{2}$ with $k=\frac{7}{2}$	A1	1.1b
			(4 marks)

Notes:

M1: Attempts to divide each term by t or alternatively multiply each term by t^{-1}

M1: Integrates each term and knows $\int \frac{1}{t}\,\mathrm{d}t=\ln t$. The + c is not required for this mark

M1: Substitutes in both limits, subtracts and sets equal to ln7

A1: Proceeds to $a=\ln\frac{7}{2}$ and states $k=\frac{7}{2}$ or exact equivalent such as 3.5

14.

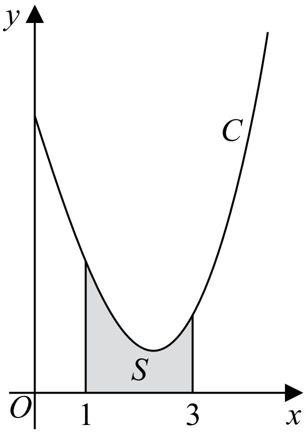

Figure 4

Figure 4 shows a sketch of part of the curve C with equation

$$y = \frac{x^2 \ln x}{3} - 2x + 5, \quad x > 0$$

The finite region S, shown shaded in Figure 4, is bounded by the curve C, the line with equation $x = 1$, the x-axis and the line with equation $x = 3$

The table below shows corresponding values of x and y with the values of y given to 4 decimal places as appropriate.

x	1	1.5	2	2.5	3
y	3	2.3041	1.9242	1.9089	2.2958

(a) Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of S, giving your answer to 3 decimal places.

(3)

(b) Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of S.

(1)

(c) Show that the exact area of S can be written in the form $\frac{a}{b} + \ln c$, where a, b and c are integers to be found.

(6)

(In part c, solutions based entirely on graphical or numerical methods are not acceptable.)

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Question	Scheme	Marks	AOs
14(a)	Uses or implies $h = 0.5$	B1	1.1b
	For correct form of the trapezium rule =	M1	1.1b
	$\frac{0.5}{2}\{3+2.2958+2(2.3041+1.9242+1.9089)\}=4.393$	A1	1.1b
		(3)	
(b)	Any valid statement reason, for example • Increase the number of strips • Decrease the width of the strips • Use more trapezia	B1	2.4
		(1)	
(c)	For integration by parts on $\int x^2 \ln x \,\mathrm{d}x$	M1	2.1
	$=\frac{x^3}{3}\ln x-\int\frac{x^2}{3}\,\mathrm{d}x$	A1	1.1b
	$\int -2x+5\,\mathrm{d}x=-x^2+5x \quad (+c)$	B1	1.1b
	All integration attempted and limits used Area of $S=\int_1^3 \frac{x^2\ln x}{3}-2x+5\,\mathrm{d}x=\left[\frac{x^3}{9}\ln x-\frac{x^3}{27}-x^2+5x\right]_{x=1}^{x=3}$	M1	2.1
	Uses correct ln laws, simplifies and writes in required form	M1	2.1
	Area of $S=\frac{28}{27}+\ln 27 \quad (a=28, b=27, c=27)$	A1	1.1b
		(6)	
			(10 marks)

Question 14 continued
Notes:
(a) **B1:** States or uses the strip width $h = 0.5$. This can be implied by the sight of $\frac{0.5}{2}\{...\}$ in the trapezium rule **M1:** For the correct form of the bracket in the trapezium rule. Must be y values rather than x values $\{\text{first } y \text{ value} + \text{last } y \text{ value} + 2\times(\text{sum of other } y \text{ values})\}$ **A1:** 4.393 **(b)** **B1:** See scheme **(c)** **M1:** Uses integration by parts the right way around. Look for $\int x^2 \ln x \, dx = Ax^3 \ln x - \int Bx^2 \, dx$ **A1:** $\frac{x^3}{3}\ln x - \int \frac{x^2}{3} \, dx$ **B1:** Integrates the $-2x+5$ term correctly $= -x^2 + 5x$ **M1:** All integration completed and limits used **M1:** Simplifies using ln law(s) to a form $\frac{a}{b} + \ln c$ **A1:** Correct answer only $\frac{28}{27} + \ln 27$

16. (a) Express $\dfrac{1}{P(11-2P)}$ in partial fractions.

(3)

A population of meerkats is being studied.

The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{22}P(11 - 2P), \quad t \geqslant 0, \qquad 0 < P < 5.5$$

where P, in thousands, is the population of meerkats and t is the time measured in years since the study began.

Given that there were 1000 meerkats in the population when the study began,

(b) determine the time taken, in years, for this population of meerkats to double,

(6)

(c) show that

$$P = \frac{A}{B + Ce^{-\frac{1}{2}t}}$$

where A, B and C are integers to be found.

(3)

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Question	Scheme	Marks	AOs
16(a)	Sets $\dfrac{1}{P(11-2P)}=\dfrac{A}{P}+\dfrac{B}{(11-2P)}$	B1	1.1a
	Substitutes either $P=0$ or $P=\dfrac{11}{2}$ into $1=A(11-2P)+BP\Rightarrow A\ or\ B$	M1	1.1b
	$\dfrac{1}{P(11-2P)}=\dfrac{1/11}{P}+\dfrac{2/11}{(11-2P)}$	A1	1.1b
		(3)	
(b)	Separates the variables $\displaystyle\int\frac{22}{P(11-2P)}\,\mathrm{d}P=\int 1\,\mathrm{d}t$	B1	3.1a
	Uses (a) and attempts to integrate $\displaystyle\int\frac{2}{P}+\frac{4}{(11-2P)}\,\mathrm{d}P=t+c$	M1	1.1b
	$2\ln P-2\ln(11-2P)=t+c$	A1	1.1b
	Substitutes $t=0,P=1\Rightarrow t=0,P=1\Rightarrow\ c=(-2\ln 9)$	M1	3.1a
	Substitutes $P=2\Rightarrow t=2\ln 2+2\ln 9-2\ln 7$	M1	3.1a
	Time $=1.89$ years	A1	3.2a
		(6)	
(c)	Uses ln laws $2\ln P-2\ln(11-2P)=t-2\ln 9$ $\Rightarrow\ln\left(\dfrac{9P}{11-2P}\right)=\dfrac{1}{2}t$	M1	2.1
	Makes '*P*' the subject $\Rightarrow\left(\dfrac{9P}{11-2P}\right)=\mathrm{e}^{\frac{1}{2}t}$ $\Rightarrow 9P=(11-2P)\mathrm{e}^{\frac{1}{2}t}$ $\Rightarrow P=\mathrm{f}\left(\mathrm{e}^{\frac{1}{2}t}\right)$ or $\Rightarrow P=\mathrm{f}\left(\mathrm{e}^{-\frac{1}{2}t}\right)$	M1	2.1
	$\Rightarrow P=\dfrac{11}{2+9\mathrm{e}^{-\frac{1}{2}t}}\Rightarrow A=11,B=2,C=9$	A1	1.1b
		(3)	
		(12 marks)	

Question 16 continued

Notes:

(a)

B1: Sets $\dfrac{1}{P(11-2P)} = \dfrac{A}{P} + \dfrac{B}{(11-2P)}$

M1: Substitutes $P = 0$ or $P = \dfrac{11}{2}$ into $1 = A(11-2P) + BP \Rightarrow A \text{ or } B$

Alternatively compares terms to set up and solve two simultaneous equations in A and B

A1: $\dfrac{1}{P(11-2P)} = \dfrac{1/11}{P} + \dfrac{2/11}{(11-2P)}$ or equivalent $\dfrac{1}{P(11-2P)} = \dfrac{1}{11P} + \dfrac{2}{11(11-2P)}$

Note: The correct answer with no working scores all three marks.

(b)

B1: Separates the variables to reach $\displaystyle\int \frac{22}{P(11-2P)} \mathrm{d}P = \int 1 \mathrm{d}t$ or equivalent

M1: Uses part (a) and $\displaystyle\int \frac{A}{P} + \frac{B}{(11-2P)} \mathrm{d}P = A \ln P \pm C \ln(11-2P)$

A1: Integrates both sides to form a correct equation including a 'c' Eg $2\ln P - 2\ln(11-2P) = t + c$

M1: Substitutes $t = 0$ and $P = 1$ to find c

M1: Substitutes $P = 2$ to find t. This is dependent upon having scored both previous M's

A1: Time = 1.89 years

(c)

M1: Uses correct log laws to move from $2\ln P - 2\ln(11-2P) = t + c$ to $\ln\left(\dfrac{P}{11-2P}\right) = \dfrac{1}{2}t + d$ for their numerical 'c'

M1: Uses a correct method to get P in terms of $\mathrm{e}^{\frac{1}{2}t}$

This can be achieved from $\ln\left(\dfrac{P}{11-2P}\right) = \dfrac{1}{2}t + d \Rightarrow \dfrac{P}{11-2P} = \mathrm{e}^{\frac{1}{2}t+d}$ followed by cross multiplication and collection of terms in P (See scheme)

Alternatively uses a correct method to get P in terms of $\mathrm{e}^{-\frac{1}{2}t}$ For example

$\dfrac{P}{11-2P} = \mathrm{e}^{\frac{1}{2}t+d} \Rightarrow \dfrac{11-2P}{P} = \mathrm{e}^{-\left(\frac{1}{2}t+d\right)} \Rightarrow \dfrac{11}{P} - 2 = \mathrm{e}^{-\left(\frac{1}{2}t+d\right)} \Rightarrow \dfrac{11}{P} = 2 + \mathrm{e}^{-\left(\frac{1}{2}t+d\right)}$ followed by division

A1: Achieves the correct answer in the form required. $P = \dfrac{11}{2 + 9\mathrm{e}^{-\frac{1}{2}t}} \Rightarrow A = 11, B = 2, C = 9$ oe

Answer ALL questions. Write your answers in the spaces provided.

1.

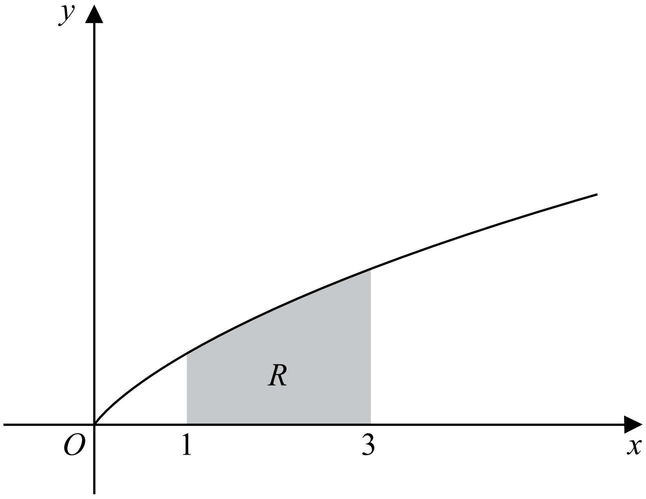

Figure 1

Figure 1 shows a sketch of the curve with equation $y = \dfrac{x}{1 + \sqrt{x}}, \quad x \geqslant 0$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the line with equation $x = 1$, the x-axis and the line with equation $x = 3$

The table below shows corresponding values of x and y for $y = \dfrac{x}{1 + \sqrt{x}}$

x	1	1.5	2	2.5	3
y	0.5	0.6742	0.8284	0.9686	1.0981

(a) Use the trapezium rule, with all the values of y in the table, to find an estimate for the area of R, giving your answer to 3 decimal places.

(3)

(b) Explain how the trapezium rule can be used to give a better approximation for the area of R.

(1)

(c) Giving your answer to 3 decimal places in each case, use your answer to part (a) to deduce an estimate for

(i) $\displaystyle\int_1^3 \frac{5x}{1 + \sqrt{x}}\,\mathrm{d}x$

(ii) $\displaystyle\int_1^3 \left(6 + \frac{x}{1 + \sqrt{x}}\right)\mathrm{d}x$

(2)

9MA0/01: Pure Mathematics Paper 1 Mark scheme

Question	Scheme	Marks	AOs
1 (a)	$\text{Area}(R) \approx \frac{1}{2} \times 0.5 \times \left[\underline{0.5 + 2(0.6742 + 0.8284 + 0.9686) + 1.0981}\right]$	B1	1.1b
		<u>M1</u>	1.1b
	$\left\{= \frac{1}{4} \times 6.5405 \;=\; 1.635125\right\} = 1.635$ (3 dp)	A1	1.1b
		(3)	
(b)	Any valid reason, for example • Increase the number of strips • Decrease the width of the strips • Use more trapezia between $x = 1$ and $x = 3$	B1	2.4
		(1)	
(c)(i)	$\left\{\int_1^3 \frac{5x}{1+\sqrt{x}}\,dx\right\} = 5(\text{"}1.635\text{"}) = 8.175$	B1ft	2.2a
(c)(ii)	$\left\{\int_1^3 \left(6 + \frac{x}{1+\sqrt{x}}\right)dx\right\} = 6(2) + (\text{"}1.635\text{"}) = 13.635$	B1ft	2.2a
		(2)	
			(6 marks)

Question 1 Notes:

(a)	
B1:	Outside brackets $\frac{1}{2} \times 0.5$ or $\frac{0.5}{2}$ or 0.25 or $\frac{1}{4}$
M1:	For structure of trapezium rule $\left[\underline{\;............\;} \right]$. No errors are allowed, e.g. an omission of a *y*-ordinate or an extra *y*-ordinate or a repeated *y*-ordinate.
A1:	Correct method leading to a correct answer only of 1.635
(b)	
B1:	See scheme
(c)	
B1:	8.175 or a value which is 5 × their answer to part (a) **Note:** Allow B1ft for 8.176 (to 3 dp) which is found from $5(1.63125) = 8.175625$ **Note:** Do not allow an answer of 8.1886… which is found directly from integration
(d)	
B1:	13.635 or a value which is 12 + their answer to part (a) **Note:** Do not allow an answer of 13.6377… which is found directly from integration

7.

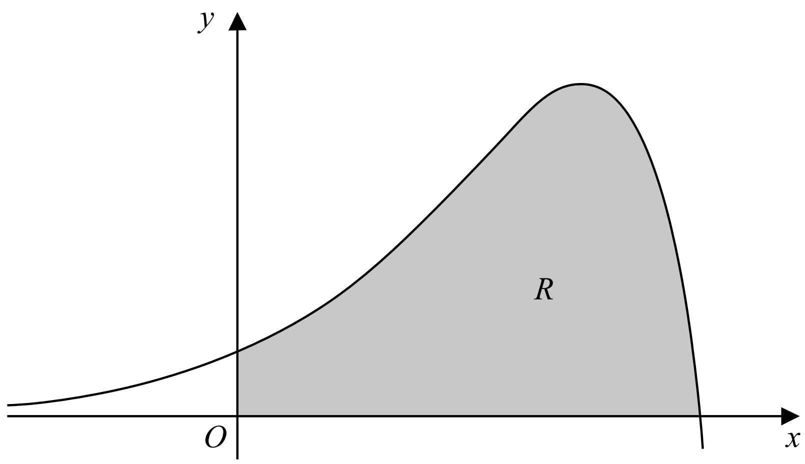

Figure 4

Figure 4 shows a sketch of part of the curve with equation

$$y = 2e^{2x} - xe^{2x}, \qquad x \in \mathbb{R}$$

The finite region R, shown shaded in Figure 4, is bounded by the curve, the x-axis and the y-axis.

Use calculus to show that the exact area of R can be written in the form $pe^4 + q$, where p and q are rational constants to be found.
(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

Question	Scheme	Marks	AOs
7	$\left\{\int xe^{2x}\,dx\right\}, \quad \left\{\begin{array}{ll} u = x & \Rightarrow \frac{du}{dx} = 1 \\ \frac{dv}{dx} = e^{2x} & \Rightarrow v = \frac{1}{2}e^{2x} \end{array}\right\}$		
	$\left\{\int xe^{2x}\,dx\right\} = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\{dx\}$	M1	3.1a
	$\left\{\int 2e^{2x} - xe^{2x}\,dx\right\} = e^{2x} - \left(\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\{dx\}\right)$	M1	1.1b
	$= e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right)$	A1	1.1b
	$\text{Area}(R) = \int_0^2 2e^{2x} - xe^{2x}\,dx = \left[\frac{5}{4}e^{2x} - \frac{1}{2}xe^{2x}\right]_0^2$	M1	2.2a
	$= \left(\frac{5}{4}e^4 - e^4\right) - \left(\frac{5}{4}e^{2(0)} - \frac{1}{2}(0)e^0\right) = \frac{1}{4}e^4 - \frac{5}{4}$	A1	2.1
		(5)	
7 Alt 1	$\left\{\int 2e^{2x} - xe^{2x}\,dx = \int (2-x)e^{2x}\,dx\right\}, \quad \left\{\begin{array}{ll} u = 2-x & \Rightarrow \frac{du}{dx} = -1 \\ \frac{dv}{dx} = e^{2x} & \Rightarrow v = \frac{1}{2}e^{2x} \end{array}\right\}$		
	$= \frac{1}{2}(2-x)e^{2x} - \int -\frac{1}{2}e^{2x}\{dx\}$	M1 M1	3.1a 1.1b
	$= \frac{1}{2}(2-x)e^{2x} + \frac{1}{4}e^{2x}$	A1	1.1b
	$\left\{\text{Area}(R) = \int_0^2 (2-x)e^{2x}\,dx = \right\} \left[\frac{1}{2}(2-x)e^{2x} + \frac{1}{4}e^{2x}\right]_0^2$	M1	2.2a
	$= \left(0 + \frac{1}{4}e^4\right) - \left(\frac{1}{2}(2)e^0 + \frac{1}{4}e^0\right) = \frac{1}{4}e^4 - \frac{5}{4}$	A1	2.1
		(5)	
			(5 marks)

Question 7 Notes:	
M1:	Attempts to solve the problem by recognising the need to apply a method of integration by parts on either xe^{2x} or $(2-x)e^{2x}$. Allow this mark for either • $\pm xe^{2x} \to \pm\lambda xe^{2x} \pm \int \mu e^{2x}\{dx\}$ • $(2-x)e^{2x} \to \pm\lambda(2-x)e^{2x} \pm \int \mu e^{2x}\{dx\}$ where $\lambda, \mu \neq 0$ are constants.
M1:	For either • $2e^{2x} - xe^{2x} \to e^{2x} \pm \frac{1}{2}xe^{2x} \pm \int \frac{1}{2}e^{2x}\{dx\}$ • $(2-x)e^{2x} \to \pm\frac{1}{2}(2-x)e^{2x} \pm \int \frac{1}{2}e^{2x}\{dx\}$
A1:	Correct integration which can be simplified or un-simplified. E.g. • $2e^{2x} - xe^{2x} \to e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right)$ • $2e^{2x} - xe^{2x} \to e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x}$ • $2e^{2x} - xe^{2x} \to \frac{5}{4}e^{2x} - \frac{1}{2}xe^{2x}$ • $(2-x)e^{2x} \to \frac{1}{2}(2-x)e^{2x} + \frac{1}{4}e^{2x}$
M1:	Deduces that the upper limit is 2 and uses limits of 2 and 0 on their integrated function
A1:	Correct proof leading to $pe^4 + q$, where $p = \frac{1}{4}$, $q = -\frac{5}{4}$

12. Show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos\theta}\ \mathrm{d}\theta = 2 - 2\ln 2$$

(7)

Question	Scheme	Marks	AOs
12	$\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos\theta}\,\mathrm{d}\theta$		
	Attempts this question by applying the substitution $u = 1+\cos\theta$ and progresses as far as achieving $\displaystyle\int ...\frac{(u-1)}{u}...$	M1	3.1a
	$u = 1+\cos\theta \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}\theta} = -\sin\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$	M1	1.1b
	$\left\{\displaystyle\int \frac{\sin 2\theta}{1+\cos\theta}\,\mathrm{d}\theta = \right\} \displaystyle\int \frac{2\sin\theta\cos\theta}{1+\cos\theta}\,\mathrm{d}\theta = \int \frac{-2(u-1)}{u}\,\mathrm{d}u$	A1	2.1
	$-2\displaystyle\int\left(1-\frac{1}{u}\right)\mathrm{d}u = -2(u - \ln u)$	M1	1.1b
		M1	1.1b
	$\left\{\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos\theta}\,\mathrm{d}\theta = \right\} = -2\left[u - \ln u\right]_2^1 = -2((1-\ln 1)-(2-\ln 2))$	M1	1.1b
	$= -2(-1+\ln 2) = 2-2\ln 2$ *	A1*	2.1
		(7)	
12 **Alt 1**	Attempts this question by applying the substitution $u = \cos\theta$ and progresses as far as achieving $\displaystyle\int ...\frac{u}{u+1}...$	M1	3.1a
	$u = \cos\theta \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}\theta} = -\sin\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$	M1	1.1b
	$\left\{\displaystyle\int \frac{\sin 2\theta}{1+\cos\theta}\,\mathrm{d}\theta = \right\} \displaystyle\int \frac{2\sin\theta\cos\theta}{1+\cos\theta}\,\mathrm{d}\theta = \int \frac{-2u}{u+1}\,\mathrm{d}u$	A1	2.1
	$\left\{= -2\displaystyle\int \frac{(u+1)-1}{u+1}\,\mathrm{d}u = -2\int 1-\frac{1}{u+1}\,\mathrm{d}u\right\} = -2(u - \ln(u+1))$	M1	1.1b
		M1	1.1b
	$\left\{\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos\theta}\,\mathrm{d}\theta = \right\} = -2\left[u - \ln(u+1)\right]_1^0 = -2((0-\ln 1)-(1-\ln 2))$	M1	1.1b
	$= -2(-1+\ln 2) = 2-2\ln 2$ *	A1*	2.1
		(7)	
		(7 marks)	

Question 12 Notes:	
M1:	See scheme
M1:	Attempts to differentiate $u = 1 + \cos\theta$ to give $\frac{du}{d\theta} = ...$ and applies $\sin 2\theta = 2\sin\theta\cos\theta$
A1:	Applies $u = 1 + \cos\theta$ to show that the integral becomes $\int \frac{-2(u-1)}{u}\, du$
M1:	Achieves an expression in u that can be directly integrated (e.g. dividing each term by u or applying partial fractions) and integrates to give an expression in u of the form $\pm\lambda u \pm \mu \ln u,\ \lambda, \mu \neq 0$
M1:	For integration in u of the form $\pm 2(u - \ln u)$
M1:	Applies u-limits of 1 and 2 to an expression of the form $\pm\lambda u \pm \mu \ln u,\ \lambda, \mu \neq 0$ and subtracts either way round
A1*:	Applies u-limits the right way round, i.e. • $\int_2^1 \frac{-2(u-1)}{u}\, du = -2\int_2^1 \left(1 - \frac{1}{u}\right) du = -2\left[u - \ln u\right]_2^1 = -2((1 - \ln 1) - (2 - \ln 2))$ • $\int_2^1 \frac{-2(u-1)}{u}\, du = 2\int_1^2 \left(1 - \frac{1}{u}\right) du = 2\left[u - \ln u\right]_1^2 = 2((2 - \ln 2) - (1 - \ln 1))$ and correctly proves $\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1 + \cos\theta}\, d\theta = 2 - 2\ln 2$, with no errors seen
Alt 1	
M1:	See scheme
M1:	Attempts to differentiate $u = \cos\theta$ to give $\frac{du}{d\theta} = ...$ and applies $\sin 2\theta = 2\sin\theta\cos\theta$
A1:	Applies $u = \cos\theta$ to show that the integral becomes $\int \frac{-2u}{u+1}\, du$
M1:	Achieves an expression in u that can be directly integrated (e.g. by applying partial fractions or a substitution $v = u+1$) and integrates to give an expression in u of the form $\pm\lambda u \pm \mu \ln(u+1),\ \lambda, \mu \neq 0$ or $\pm\lambda v \pm \mu \ln v,\ \lambda, \mu \neq 0$, where $v = u+1$
M1:	For integration in u in the form $\pm 2(u - \ln(u+1))$
M1:	Either • Applies u-limits of 0 and 1 to an expression of the form $\pm\lambda u \pm \mu \ln(u+1),\ \lambda, \mu \neq 0$ and subtracts either way round • Applies v-limits of 1 and 2 to an expression of the form $\pm\lambda v \pm \mu \ln v,\ \lambda, \mu \neq 0$, where $v = u+1$ and subtracts either way round
A1*:	Applies u-limits the right way round, (o.e. in v) i.e. • $\int_1^0 \frac{-2u}{u+1}\, du = -2\int_1^0 \left(1 - \frac{1}{u+1}\right) du = -2\left[u - \ln(u+1)\right]_1^0 = -2((0 - \ln 1) - (1 - \ln 2))$ • $\int_1^0 \frac{-2u}{u+1}\, du = 2\int_0^1 \left(1 - \frac{1}{u+1}\right) du = 2\left[u - \ln(u+1)\right]_0^1 = 2((1 - \ln 2) - (0 - \ln 1))$ and correctly proves $\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1 + \cos\theta}\, d\theta = 2 - 2\ln 2$, with no errors seen

10.

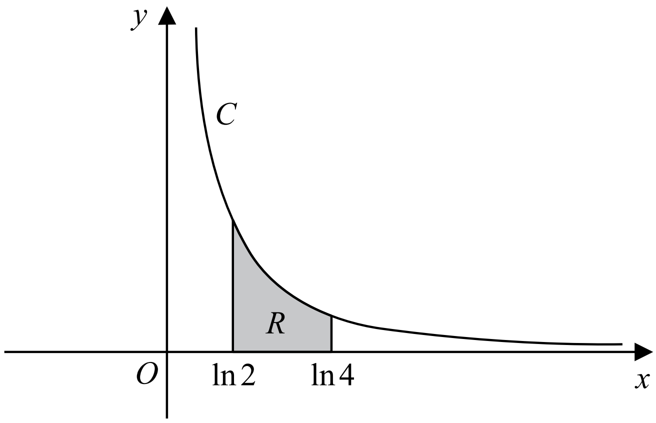

Figure 4

Figure 4 shows a sketch of the curve C with parametric equations

$$x = \ln(t + 2), \quad y = \frac{1}{t+1}, \qquad t > -\frac{2}{3}$$

(a) State the domain of values of x for the curve C.

(1)

The finite region R, shown shaded in Figure 4, is bounded by the curve C, the line with equation $x = \ln 2$, the x-axis and the line with equation $x = \ln 4$

(b) Use calculus to show that the area of R is $\ln\left(\frac{3}{2}\right)$.

(8)

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Question	Scheme	Marks	AOs
10 (a)	$x > \ln\left(\frac{4}{3}\right)$	B1	2.2a
		(1)	
(b)	Attempts to apply $\int y \frac{dx}{dt} dt$	M1	3.1a
	$\left\{\int y \frac{dx}{dt} dt = \right\} = \int \left(\frac{1}{t+1}\right)\left(\frac{1}{t+2}\right) dt$	A1	1.1b
	$\frac{1}{(t+1)(t+2)} \equiv \frac{A}{(t+1)} + \frac{B}{(t+2)} \Rightarrow 1 \equiv A(t+2) + B(t+1)$	M1	3.1a
	$\{A = 1, B = -1 \Rightarrow\}$ gives $\frac{1}{(t+1)} - \frac{1}{(t+2)}$	A1	1.1b
	$\left\{\int \left(\frac{1}{(t+1)} - \frac{1}{(t+2)}\right) dt = \right\} \ln(t+1) - \ln(t+2)$	M1	1.1b
		A1	1.1b
	$\text{Area}(R) = [\ln(t+1) - \ln(t+2)]_0^2 = (\ln 3 - \ln 4) - (\ln 1 - \ln 2)$	M1	2.2a
	$= \ln 3 - \ln 4 + \ln 2 = \ln\left(\frac{(3)(2)}{4}\right) = \ln\left(\frac{6}{4}\right)$		
	$= \ln\left(\frac{3}{2}\right)$ *	A1*	2.1
		(8)	
(b) Alt 1	Attempts to apply $\int y dx = \int \frac{1}{e^x - 2 + 1} dx = \int \frac{1}{e^x - 1} dx$, with a substitution of $u = e^x - 1$	M1	3.1a
	$\left\{\int y dx \right\} = \int \left(\frac{1}{u}\right)\left(\frac{1}{u+1}\right) du$	A1	1.1b
	$\frac{1}{u(u+1)} \equiv \frac{A}{u} + \frac{B}{(u+1)} \Rightarrow 1 \equiv A(u+1) + Bu$	M1	3.1a
	$\{A = 1, B = -1 \Rightarrow\}$ gives $\frac{1}{u} - \frac{1}{(u+1)}$	A1	1.1b
	$\left\{\int \left(\frac{1}{u} - \frac{1}{(u+1)}\right) du = \right\} \ln u - \ln(u+1)$	M1	1.1b
		A1	1.1b
	$\text{Area}(R) = [\ln u - \ln(u+1)]_1^3 = (\ln 3 - \ln 4) - (\ln 1 - \ln 2)$	M1	2.2a
	$= \ln 3 - \ln 4 + \ln 2 = \ln\left(\frac{(3)(2)}{4}\right) = \ln\left(\frac{6}{4}\right)$		
	$= \ln\left(\frac{3}{2}\right)$ *	A1 *	2.1
		(8)	
			(9 marks)

Question	Scheme	Marks	AOs
10 (b) Alt 2	Attempts to apply $\int y\mathrm{d}x = \int \frac{1}{\mathrm{e}^x - 2 + 1}\,\mathrm{d}x = \int \frac{1}{\mathrm{e}^x - 1}\,\mathrm{d}x$, with a substitution of $v = \mathrm{e}^x$	M1	3.1a
	$\left\{\int y\mathrm{d}x\right\} = \int \left(\frac{1}{v-1}\right)\left(\frac{1}{v}\right)\mathrm{d}v$	A1	1.1b
	$\frac{1}{(v-1)v} \equiv \frac{A}{(v-1)} + \frac{B}{v} \Rightarrow 1 \equiv Av + B(v-1)$	M1	3.1a
	$\{A = 1,\ B = -1 \Rightarrow\}$ gives $\frac{1}{(v-1)} - \frac{1}{v}$	A1	1.1b
	$\left\{\int \left(\frac{1}{(v-1)} - \frac{1}{v}\right)\mathrm{d}v = \right\}\ \ln(v-1) - \ln v$	M1	1.1b
		A1	1.1b
	$\text{Area}(R) = \left[\ln(v-1) - \ln v\right]_2^4 = (\ln 3 - \ln 4) - (\ln 1 - \ln 2)$	M1	2.2a
	$= \ln 3 - \ln 4 + \ln 2 = \ln\left(\frac{(3)(2)}{4}\right) = \ln\left(\frac{6}{4}\right)$		
	$= \ln\left(\frac{3}{2}\right)$ *	A1 *	2.1
		(8)	

Question 10 Notes:	
(a)	
B1:	Uses $x = \ln(t+2)$ with $t > -\frac{2}{3}$ to deduce the correct domain, $x > \ln\left(\frac{4}{3}\right)$
(b)	
M1:	Attempts to solve the problem by either • a parametric process or • a Cartesian process with a substitution of either $u = e^x - 1$ or $v = e^x$
A1:	Obtains • $\int\left(\frac{1}{t+1}\right)\left(\frac{1}{t+2}\right)dt$ from a parametric approach • $\int\left(\frac{1}{u}\right)\left(\frac{1}{u+1}\right)du$ from a Cartesian approach with $u = e^x - 1$ • $\int\left(\frac{1}{v-1}\right)\left(\frac{1}{v}\right)dv$ from a Cartesian approach with $v = e^x$
M1:	Applies a strategy of attempting to express either $\frac{1}{(t+1)(t+2)}$, $\frac{1}{u(u+1)}$ or $\frac{1}{(v-1)v}$ as partial fractions
A1:	Correct partial fractions for their method
M1:	Integrates to give either • $\pm\alpha\ln(t+1) \pm \beta\ln(t+2)$ • $\pm\alpha\ln u \pm \beta\ln(u+1)$; $\alpha, \beta \neq 0$, where $u = e^x - 1$ • $\pm\alpha\ln(v-1) \pm \beta\ln v$; $\alpha, \beta \neq 0$, where $v = e^x$
A1:	Correct integration for their method
M1:	Either • Parametric approach: Deduces and applies limits of 2 and 0 in *t* and subtracts the correct way round • Cartesian approach: Deduces and applies limits of 3 and 1 in *u*, where $u = e^x - 1$, and subtracts the correct way round • Cartesian approach: Deduces and applies limits of 4 and 2 in *v*, where $v = e^x$, and subtracts the correct way round
A1*:	Correctly shows that the area of *R* is $\ln\left(\frac{3}{2}\right)$, with no errors seen in their working

7. Given that $k \in \mathbb{Z}^+$

(a) show that $\displaystyle\int_k^{3k} \frac{2}{(3x-k)}\,dx$ is independent of k,

(4)

(b) show that $\displaystyle\int_k^{2k} \frac{2}{(2x-k)^2}\,dx$ is inversely proportional to k.

(3)

Question	Scheme	Marks	AOs
7 (a)	$\int \frac{2}{(3x-k)}\,dx = \frac{2}{3}\ln(3x-k)$	M1 A1	1.1a 1.1b
	$\int_k^{3k} \frac{2}{(3x-k)}\,dx = \frac{2}{3}\ln(9k-k) - \frac{2}{3}\ln(3k-k)$	dM1	1.1b
	$= \frac{2}{3}\ln\left(\frac{8\cancel{k}}{2\cancel{k}}\right) = \frac{2}{3}\ln 4$ oe	A1	2.1
		(4)	
(b)	$\int \frac{2}{(2x-k)^2}\,dx = -\frac{1}{(2x-k)}$	M1	1.1b
	$\int_k^{2k} \frac{2}{(2x-k)^2}\,dx = -\frac{1}{(4k-k)} + \frac{1}{(2k-k)}$	dM1	1.1b
	$= \frac{2}{3k} \quad \left(\propto \frac{1}{k}\right)$	A1	2.1
		(3)	
			(7 marks)

(a)

M1: $\int \frac{2}{(3x-k)}\,dx = A\ln(3x-k)$ Condone a missing bracket

A1: $\int \frac{2}{(3x-k)}\,dx = \frac{2}{3}\ln(3x-k)$

Allow recovery from a missing bracket if in subsequent work $A\ln 9k - k \rightarrow A\ln 8k$

dM1: For substituting k and $3k$ into their $A\ln(3x-k)$ and subtracting either way around

A1: Uses correct ln work and notation to show that I $= \frac{2}{3}\ln\left(\frac{8}{2}\right)$ or $\frac{2}{3}\ln 4$ oe (ie independent of k)

(b)

M1: $\int \frac{2}{(2x-k)^2}\,dx = \frac{C}{(2x-k)}$

dM1: For substituting k and $2k$ into their $\frac{C}{(2x-k)}$ and subtracting

A1: Shows that it is inversely proportional to k Eg proceeds to the answer is of the form A/k with $A = 2/3$

There is no need to perform the whole calculation. Accept from $-\frac{1}{(3k)} + \frac{1}{(k)} = \left(-\frac{1}{3}+1\right) \times \frac{1}{k} \propto \frac{1}{k}$

If the calculation is performed it must be correct.

Do not isw here. They should know when they have an expression that is inversely proportional to *k*.

You may see substitution used but the mark is scored for the same result. See below

$u = 2x - k \rightarrow \left[\frac{C}{u}\right]$ for M1 with limits $3k$ and k used for dM1

10. The height above ground, H metres, of a passenger on a roller coaster can be modelled by the differential equation

$$\frac{dH}{dt} = \frac{H\cos(0.25t)}{40}$$

where t is the time, in seconds, from the start of the ride.

Given that the passenger is 5 m above the ground at the start of the ride,

(a) show that $H = 5e^{0.1\sin(0.25t)}$

(5)

(b) State the maximum height of the passenger above the ground.

(1)

The passenger reaches the maximum height, for the second time, T seconds after the start of the ride.

(c) Find the value of T.

(2)

Question	Scheme	Marks	AOs
10(a)	$\frac{dH}{dt}=\frac{H\cos 0.25t}{40} \Rightarrow \int \frac{1}{H}dH=\int \frac{\cos 0.25t}{40}dt$	M1	3.1a
	$\ln H=\frac{1}{10}\sin 0.25t(+c)$	M1 A1	1.1b 1.1b
	Substitutes $t=0, H=5 \Rightarrow c=\ln(5)$	dM1	3.4
	$\ln\left(\frac{H}{5}\right)=\frac{1}{10}\sin 0.25t \Rightarrow H=5e^{0.1\sin 0.25t}$ *	A1*	2.1
		(5)	
(b)	Max height = $5e^{0.1}=5.53$ m (Condone lack of units)	B1	3.4
		(1)	
(c)	Sets $0.25t=\frac{5\pi}{2}$	M1	3.1b
	31.4	A1	1.1b
		(2)	
			(8 marks)

(a)

M1: Separates the variables to reach $\int \frac{1}{H}dH=\int \frac{\cos 0.25t}{40}dt$ or equivalent.

The integral signs need to be present on both sides and the dH AND dt need to be in the correct positions.

M1: Integrates both sides to reach $\ln H = A\sin 0.25t$ or equivalent with or without the + c

A1: $\ln H=\frac{1}{10}\sin 0.25t+c$ or equivalent with or without the + c. Allow two constants, one either side

If the 40 was on the lhs look for $40\ln H=4\sin 0.25t+c$ or equivalent.

dM1: Substitutes $t=0, H=5 \Rightarrow c=..$ There needs to have been a single "+ c" to find.

It is dependent upon the previous M mark. You may allow even if you don't explicitly see "$t=0, H=5$" as it may be implied from their previous line

If the candidate has attempted to change the subject and made an error/ slip then condone it for this M but not the final A. Eg. $40\ln H=4\sin 0.25t+c \Rightarrow\Rightarrow H^{40}=e^{4\sin 0.25t}+e^{c} \Rightarrow 5^{40}=1+e^{c} \Rightarrow c=...$

Also many students will be attempting to get to the given answer so condone the method of finding c =...
These students will lose the A1* mark

A1*: Proceeds via $\ln H=\frac{1}{10}\sin 0.25t+\ln 5$ or equivalent to the given answer $H=5e^{0.1\sin 0.25t}$ with at least one correct intermediate line and no incorrect work.

DO NOT condone c's going to c's when they should be e^{c} or A

Accept as a minimum $\ln H = \frac{1}{10}\sin 0.25t + \ln 5 \Rightarrow H = e^{\frac{1}{10}\sin 0.25t + \ln 5}$ or $H = e^{\frac{1}{10}\sin 0.25t} \times e^{+\ln 5}$ before sight of the given answer

If the only error was to omit the integration signs on line 1, thus losing the first M1, allow the candidate to have access to this mark following a correct intermediate line (see above).

If they attempt to change the subject first then the constant of integration must have been adapted if the A1* is to be awarded. $\ln H = \frac{1}{10}\sin 0.25t + c \Rightarrow H = e^{\frac{1}{10}\sin 0.25t + c} \Rightarrow H = Ae^{\frac{1}{10}\sin 0.25t}$

The dM1 and A1* under this method are awarded at virtually the same time.

Also, for the final two marks, you may see a proof from $\int_0^H \frac{40}{H}\, dH = \int_5^t \cos 0.25t\, dt$

………

There is an alternative via the use of an integrating factor.

………

(b)

B1: States that the maximum height is 5.53 m   Accept $5e^{0.1}$ Condone a lack of units here, but penalise if incorrect units are used.

(c)

M1: For identifying that it would reach the maximum height for the 2nd time when $0.25t = \frac{5\pi}{2}$ or 450

A1: Accept awrt 31.4 or 10π Allow if units are seen

13. Show that

$$\int_0^2 2x\sqrt{x+2}\,\mathrm{d}x = \frac{32}{15}\left(2+\sqrt{2}\right)$$

(7)

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Question	Scheme for Substitution		Marks	AOs
13	Chooses a suitable method for $\int_0^2 2x\sqrt{x+2}\,dx$ Award for • Using a valid substitution u =..., changing the terms to u's • integrating and using appropriate limts .		M1	3.1a
	Substitution $u=\sqrt{x+2}\Rightarrow \frac{dx}{du}=2u$ oe	Substitution $u=x+2\Rightarrow \frac{dx}{du}=1$ oe	B1	1.1b
	$\int 2x\sqrt{x+2}\,dx$ $=\int A(u^2\pm2)u^2 du$	$\int 2x\sqrt{x+2}\,dx$ $=\int A(u\pm2)\sqrt{u}du$	M1	1.1b
	$=Pu^5\pm Qu^3$	$=Su^{\frac{5}{2}}\pm Tu^{\frac{3}{2}}$	dM1	2.1
	$=\frac{4}{5}u^5-\frac{8}{3}u^3$	$=\frac{4}{5}u^{\frac{5}{2}}-\frac{8}{3}u^{\frac{3}{2}}$	A1	1.1b
	Uses limits 2 and $\sqrt{2}$ the correct way around	Uses limits 4 and 2 the correct way around	ddM1	1.1b
	$=\frac{32}{15}\left(2+\sqrt{2}\right)$ *		A1*	2.1
			(7)	
				(7 marks)

M1: For attempting to integrate using substitution. Look for

- terms and limits changed to u's. Condone slips and errors/omissions on changing $dx\to du$
- attempted multiplication of terms and raising of at least one power of u by one. Condone slips
- Use of at least the top correct limit. For instance if they go back to x's the limit is 2

B1: For substitution it is for giving the substitution and stating a correct $\frac{dx}{du}$

Eg, $u=\sqrt{x+2}\Rightarrow \frac{dx}{du}=2u$ or equivalent such as $\frac{du}{dx}=\frac{1}{2\sqrt{x+2}}$

M1: It is for attempting to get all aspects of the integral in terms of 'u'.
All terms must be attempted including the dx. You are only condoning slips on signs and coefficients

dM1: It is for using a correct method of expanding and integrating each term (seen at least once) . It is dependent upon the previous M

A1: Correct answer in x or u See scheme

ddM1: Dependent upon the previous M, it is for using the correct limits for their integral, **the correct way around**

A1*: Proceeds correctly to $=\frac{32}{15}\left(2+\sqrt{2}\right)$. **Note that this is a given answer**

There must be at one least correct intermediate line between $\left[\frac{4}{5}u^5-\frac{8}{3}u^3\right]_{\sqrt{2}}^{2}$ and $\frac{32}{15}\left(2+\sqrt{2}\right)$

Question Alt	Scheme for by parts	Marks	AOs
13	Chooses a suitable method for $\int_0^2 2x\sqrt{x+2}\,dx$ Award for • using by parts the correct way around • and using limits	M1	3.1a
	$\int \left(\sqrt{x+2}\right)dx = \frac{2}{3}(x+2)^{\frac{3}{2}}$	B1	1.1b
	$\int 2x\sqrt{x+2}\,dx = Ax(x+2)^{\frac{3}{2}} - \int B(x+2)^{\frac{3}{2}}(dx)$	M1	1.1b
	$= Ax(x+2)^{\frac{3}{2}} - C(x+2)^{\frac{5}{2}}$	dM1	2.1
	$= \frac{4}{3}x(x+2)^{\frac{3}{2}} - \frac{8}{15}(x+2)^{\frac{5}{2}}$	A1	1.1b
	Uses limits 2 and 0 the correct way around	ddM1	1.1b
	$= \frac{32}{15}\left(2+\sqrt{2}\right)$	A1*	2.1
		(7)	

M1: For attempting using by parts to solve It is a problem- solving mark and all elements do not have to be correct.

- the formula applied the correct way around. You may condone incorrect attempts at integrating $\sqrt{x+2}$ for this problem solving mark
- further integration, again, this may not be correct, and the use of at least the top limit of 2

B1: For $\int \left(\sqrt{x+2}\right)dx = \frac{2}{3}(x+2)^{\frac{3}{2}}$ oe May be awarded $\int_0^2 2x\sqrt{x+2}\,dx \to x^2 \times \frac{2(x+2)^{\frac{3}{2}}}{3}$

M1: For integration by parts the right way around. Award for $Ax(x+2)^{\frac{3}{2}} - \int B(x+2)^{\frac{3}{2}}(dx)$

dM1: For integrating a second time. Award for $Ax(x+2)^{\frac{3}{2}} - C(x+2)^{\frac{5}{2}}$

A1: $\frac{4}{3}x(x+2)^{\frac{3}{2}} - \frac{8}{15}(x+2)^{\frac{5}{2}}$ which may be un simplified

ddM1: Dependent upon the previous M, it is for using the limits 2 and 0 the **correct way around**

A1*: Proceeds to $= \frac{32}{15}\left(2+\sqrt{2}\right)$. **Note that this is a given answer**.

At least one correct intermediate line must be seen. (See substitution). You would condone missing dx's

10. A spherical mint of radius 5 mm is placed in the mouth and sucked.
Four minutes later, the radius of the mint is 3 mm.

In a simple model, the rate of decrease of the radius of the mint is inversely proportional to the square of the radius.

Using this model and all the information given,

(a) find an equation linking the radius of the mint and the time.
(You should define the variables that you use.)

(5)

(b) Hence find the total time taken for the mint to completely dissolve. Give your answer in minutes and seconds to the nearest second.

(2)

(c) Suggest a limitation of the model.

(1)

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Question	Scheme		Marks	AOs
10 (a)	$\frac{\mathrm{d}r}{\mathrm{d}t} \propto \pm\frac{1}{r^2}$ or $\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\frac{k}{r^2}$ (for k or a numerical k)		M1	3.3
	$\int r^2\,\mathrm{d}r = \int \pm k\,\mathrm{d}t \Rightarrow ...$ (for k or a numerical k)		M1	2.1
	$\frac{1}{3}r^3 = \pm kt\ \{+c\}$		A1	1.1b
	$t=0,\ r=5$ and $t=4,\ r=3$	$t=0,\ r=5$ and $t=240,\ r=3$	M1	3.1a
	gives $\frac{1}{3}r^3 = -\frac{49}{6}t + \frac{125}{3}$, where r, in mm, is the radius {of the mint} and t, in minutes, is the time from when it {the mint} was placed in the mouth	gives $\frac{1}{3}r^3 = -\frac{49}{360}t + \frac{125}{3}$, where r, in mm, is the radius {of the mint} and t, in seconds, is the time from when it {the mint} was placed in the mouth	A1	1.1b
			(5)	
(b)	$r=0 \Rightarrow 0 = -\frac{49}{6}t + \frac{125}{3} \Rightarrow\ 0 = -49t + 250\ \Rightarrow t = ...$		M1	3.4
	time = 5 minutes 6 seconds		A1	1.1b
			(2)	
(c)	Suggests a suitable limitation of the model. E.g. • Model does not consider how the mint is sucked • Model does not consider whether the mint is bitten • Model is limited for times up to 5 minutes 6 seconds, o.e. • Not valid for times greater than 5 minutes 6 seconds, o.e. • Mint may not retain the shape of a sphere (or have uniform radius) as it is being sucked • The model indicates that the radius of the mint is negative after it dissolves • Model does not consider the temperature in the mouth • Model does not consider rate of saliva production • Mint could be swallowed before it dissolves in the mouth		B1	3.5b
			(1)	
				(8 marks)

Notes for Question 10	
(a)	
M1:	Translates the description of the model into mathematics. See scheme.
M1:	Separates the variables of their differential equation which is in the form $\frac{\mathrm{d}r}{\mathrm{d}t} = \mathrm{f}(r)$ and some attempt at integration. (e.g. attempts to integrate at least one side). e.g. $\int r^2\,\mathrm{d}r = \int \pm k\,\mathrm{d}t$ and some attempt at integration. Condone the lack of integral signs
Note:	You can imply the M1 mark for $r^2 dr = -k\,\mathrm{d}t \Rightarrow \frac{1}{3}r^3 = -kt$
Note:	A numerical value of *k* (e.g. $k = \pm 1$) is allowed for the first two M marks
A1:	Correct integration to give $\frac{1}{3}r^3 = \pm kt$ with or without a constant of integration, *c*
M1:	For a complete process of using the boundary conditions to find both their unknown constants and finds an equation linking *r* and *t* So applies either • $t = 0,\ r = 5$ and $t = 4,\ r = 3$, or • $t = 0,\ r = 5$ and $t = 240,\ r = 3$, ***on their integrated equation*** to find their constants *k* and *c* ***and*** obtains an equation linking *r* and *t*
A1:	Correct equation, with variables *r* and *t* fully defined including correct reference to units. • $\frac{1}{3}r^3 = -\frac{49}{6}t + \frac{125}{3}$, {or an equivalent equation,} where *r*, in mm, is the radius {of the mint} and *t*, in minutes, is the time from when it {the mint} was placed in the mouth • $\frac{1}{3}r^3 = -\frac{49}{360}t + \frac{125}{3}$, {or an equivalent equation,} where *r*, in mm, is the radius {of the mint} and *t*, in seconds, is the time from when it {the mint} was placed in the mouth
Note:	Allow correct equations such as • in minutes, $r = \sqrt[3]{\frac{250-49t}{2}}$, $r^3 = -\frac{49}{2}t + 125$ or $t = \frac{250-2r^3}{49}$ • in seconds, $r = \sqrt[3]{\frac{15000-49t}{120}}$, $r^3 = -\frac{49}{120}t + 125$ or $t = \frac{15000-120r^3}{49}$
Note:	*t* defined as "the time from the start" is not sufficient for the final A1
(b)	
M1:	Sets $r = 0$ in their part (a) equation which links *r* with *t* and rearranges to make $t = ...$
A1:	5 minutes 6 seconds cao (**Note:** 306 seconds with no reference to 5 minutes 6 seconds is A0)
Note:	Give M0 if their equation would solve to give a negative time or a negative time is found
Note:	You can mark part (a) and part (b) together
(c)	
B1:	See scheme
Note:	Do not accept by itself • mint may not dissolve at a constant rate • rate of decrease of mint must be constant • $0 \le t < \frac{250}{49}$, $r \ge 0$; without any written explanation • reference to a mint having $r > 5$

13.

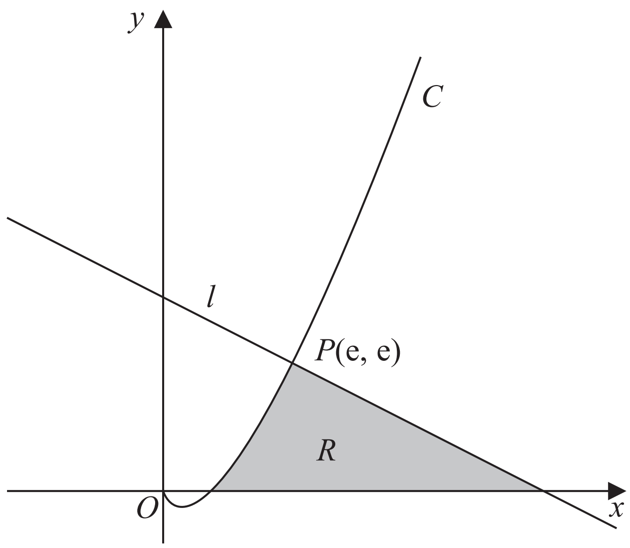

Figure 2

Figure 2 shows a sketch of part of the curve C with equation $y = x \ln x, \quad x > 0$

The line l is the normal to C at the point $P(\mathrm{e}, \mathrm{e})$

The region R, shown shaded in Figure 2, is bounded by the curve C, the line l and the x-axis.

Show that the exact area of R is $A\mathrm{e}^2 + B$ where A and B are rational numbers to be found.

(10)

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P 5 8 3 4 9 A 0 3 8 4 4

Question	Scheme	Marks	AOs
13	$C: y = x\ln x$; l is a normal to C at $P(\text{e}, \text{e})$ Let x_A be the x-coordinate of where l cuts the x-axis		
	$\frac{\text{d}y}{\text{d}x} = \ln x + x\left(\frac{1}{x}\right) \quad \{= 1 + \ln x\}$	M1 A1	2.1 1.1b
	$x = \text{e},\ m_T = 2 \Rightarrow m_N = -\frac{1}{2} \Rightarrow y - \text{e} = -\frac{1}{2}(x - \text{e})$ $y = 0 \Rightarrow -\text{e} = -\frac{1}{2}(x - \text{e}) \Rightarrow x = ...$	M1	3.1a
	l meets x-axis at $x = 3\text{e}$ (allow $x = 2\text{e} + \text{e}\ln\text{e}$)	A1	1.1b
	{Areas:} **either** $\int_1^{\text{e}} x\ln x\,\text{d}x = [\ ...\ ]_1^{\text{e}} = ...$ **or** $\frac{1}{2}((\text{their } x_A) - \text{e})\text{e}$	M1	2.1
	$\left\{\int x\ln x\,\text{d}x = \right\}\ \frac{1}{2}x^2\ln x - \int \frac{1}{x}.\left(\frac{x^2}{2}\right)\{\text{d}x\}$	M1	2.1
	$\left\{= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,\{\text{d}x\}\right\} = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$	dM1 A1	1.1b 1.1b
	$\text{Area}(R_1) = \int_1^{\text{e}} x\ln x\,\text{d}x = [\ ...\ ]_1^{\text{e}} = ...$; $\text{Area}(R_2) = \frac{1}{2}((\text{their } x_A) - \text{e})\text{e}$ and so, $\text{Area}(R) = \text{Area}(R_1) + \text{Area}(R_2) \quad \{= \tfrac{1}{4}\text{e}^2 + \tfrac{1}{4} + \text{e}^2\}$	M1	3.1a
	$\text{Area}(R) = \tfrac{5}{4}\text{e}^2 + \tfrac{1}{4}$	A1	1.1b
		(10)	

Notes for Question 13

M1:	Differentiates by using the product rule to give $\ln x + x(\text{their } \text{g}'(x))$, where $\text{g}(x) = \ln x$
A1:	Correct differentiation of $y = x\ln x$, which can be un-simplified or simplified
M1:	Complete strategy to find the x coordinate where their normal to C at $P(\text{e}, \text{e})$ meets the x-axis i.e. Sets $y = 0$ in $y - \text{e} = m_N(x - \text{e})$ to find $x = ...$
Note:	m_T is found by using calculus and $m_N \neq m_T$
A1:	l meets x-axis at $x = 3\text{e}$, allowing un-simplified values for x such as $x = 2\text{e} + \text{e}\ln\text{e}$
Note:	Allow $x = \text{awrt } 8.15$
M1:	Scored for either • Area under curve $= \int_1^{\text{e}} x\ln x\,\text{d}x = [\ ...\ ]_1^{\text{e}} = ...$, with limits of e and 1 and some attempt to substitute these and subtract • **or** Area under line $= \frac{1}{2}((\text{their } x_A) - \text{e})\text{e}$, with a valid attempt to find x_A
M1:	Integration by parts the correct way around to give $Ax^2\ln x - \int B\left(\frac{x^2}{x}\right)\{\text{d}x\}$; $A \neq 0,\ B > 0$
dM1:	**dependent on the previous M mark** Integrates the second term to give $\pm\lambda x^2$; $\lambda \neq 0$
A1:	$\frac{1}{2}x^2\ln x - \frac{1}{4}x^2$
M1:	Complete strategy of finding the area of R by finding the sum of two key areas. See scheme.
A1:	$\tfrac{5}{4}\text{e}^2 + \tfrac{1}{4}$

Notes for Question 13 Continued	
Note:	$\text{Area}(R_2)$ can also be found by integrating the line *l* between limits of e and their x_A i.e. $\text{Area}(R_2) = \int_{\text{e}}^{\text{their } x_A} \left(-\frac{1}{2}x + \frac{3}{2}\text{e}\right) \text{d}x = \left[\; ... \; \right]_{\text{e}}^{\text{their } x_A} = ...$
Note:	<u>**Calculator approach with no algebra, differentiation or integration seen:**</u> • Finding *l* cuts through the *x*-axis at awrt 8.15 is 2nd M1 2nd A1 • Finding area between curve and the *x*-axis between $x = 1$ and $x = \text{e}$ to give awrt 2.10 is 3rd M1 • Using the above information (must be seen) to apply $\text{Area}(R) = 2.0972... + 7.3890... = 9.4862...$ is final M1 Therefore, a maximum of 4 marks out of the 10 available.

13. The curve C with equation

$$y = \frac{p - 3x}{(2x - q)(x + 3)} \qquad x \in \mathbb{R}, x \neq -3, x \neq 2$$

where p and q are constants, passes through the point $\left(3, \frac{1}{2}\right)$ and has two vertical asymptotes with equations $x = 2$ and $x = -3$

(a) (i) Explain why you can deduce that $q = 4$

(ii) Show that $p = 15$

(3)

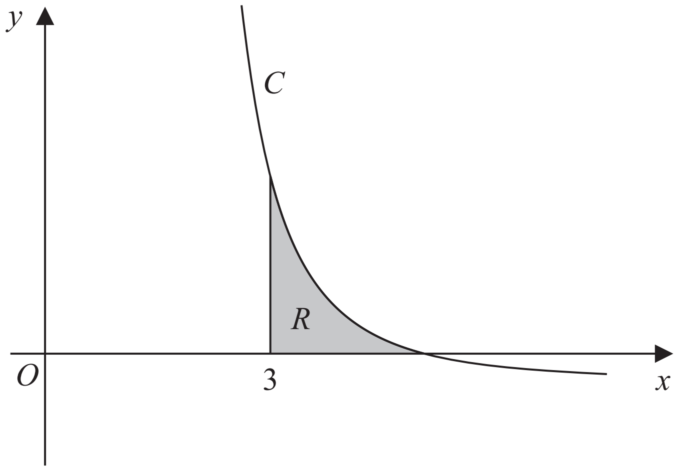


Figure 4

Figure 4 shows a sketch of part of the curve C. The region R, shown shaded in Figure 4, is bounded by the curve C, the x-axis and the line with equation $x = 3$

(b) Show that the exact value of the area of R is $a\ln 2 + b\ln 3$, where a and b are rational constants to be found.

(8)

Question	Scheme	Marks	AOs
13 (a)	(i) **Explains** $2x-q=0$ when $x=2$ oe Hence $q=4$ *	B1*	2.4
	(ii) Substitutes $\left(3,\frac{1}{2}\right)$ into $y=\frac{p-3x}{(2x-4)(x+3)}$ and solves	M1	1.1b
	$\frac{1}{2}=\frac{p-9}{(2)\times(6)} \Rightarrow p-9=6 \Rightarrow p=15$*	A1*	2.1
		(3)	
(b)	Attempts to write $\frac{15-3x}{(2x-4)(x+3)}$ in PF's and integrates using lns between 3 and another value of *x*.	M1	3.1a
	$\frac{15-3x}{(2x-4)(x+3)}=\frac{A}{(2x-4)}+\frac{B}{(x+3)}$ leading to *A* and *B*	M1	1.1b
	$\frac{15-3x}{(2x-4)(x+3)}=\frac{1.8}{(2x-4)}-\frac{2.4}{(x+3)}$ or $\frac{0.9}{(x-2)}-\frac{2.4}{(x+3)}$ oe	A1	1.1b
	I = $\int\frac{15-3x}{(2x-4)(x+3)}\mathrm{d}x=m\ln(2x-4)+n\ln(x+3)+(c)$	M1	1.1b
	I = $\int\frac{15-3x}{(2x-4)(x+3)}\mathrm{d}x=0.9\ln(2x-4)-2.4\ln(x+3)$ oe	A1ft	1.1b
	Deduces that Area Either $\int_3^5\frac{15-3x}{(2x-4)(x+3)}\mathrm{d}x$ Or $[.............]_3^5$	B1	2.2a
	Uses correct ln work seen at least once for $\ln 6=\ln 2+\ln 3$ **or** $\ln 8=3\ln 2$ $[0.9\ln(6)-2.4\ln(8)]-[0.9\ln(2)-2.4\ln(6)]$ $=3.3\ln 6-7.2\ln 2-0.9\ln 2$	dM1	2.1
	$=3.3\ln 3-4.8\ln 2$	A1	1.1b
		(8)	
			(11marks)

(a)

B1*: Is able to link $2x-q=0$ and $x=2$ to explain why $q=4$

Eg "The asymptote $x=2$ is where $2x-q=0$ so $4-q=0 \Rightarrow q=4$"

"The curve is not defined when $2\times 2-q=0 \Rightarrow q=4$"

There **must be some words** explaining why $q = 4$ and in most cases, you should see a reference to either "the asymptote $x=2$", "the curve is not defined at $x=2$", 'the denominator is 0 at $x=2$"

M1: Substitutes $\left(3,\frac{1}{2}\right)$ into $y=\frac{p-3x}{(2x-4)(x+3)}$ and solves

Alternatively substitutes $\left(3,\frac{1}{2}\right)$ into $y=\frac{15-3x}{(2x-4)(x+3)}$ and shows $\frac{1}{2}=\frac{6}{(2)\times(6)}$ oe

A1*: Full proof showing all necessary steps $\frac{1}{2}=\frac{p-9}{(2)\times(6)} \Rightarrow p-9=6 \Rightarrow p=15$

In the alternative there would have to be some recognition that these are equal eg ✓ hence $p=15$

(b)

M1: Scored for an overall attempt at using PF's and integrating with lns seen with sight of limits 3 and another value of *x*.

M1: $\frac{15-3x}{(2x-4)(x+3)}=\frac{A}{(2x-4)}+\frac{B}{(x+3)}$ leading to *A* and *B*

A1: $\frac{15-3x}{(2x-4)(x+3)}=\frac{1.8}{(2x-4)}-\frac{2.4}{(x+3)}$, or for example $\frac{0.9}{(x-2)}-\frac{2.4}{(x+3)}$, $\frac{9}{(10x-20)}-\frac{12}{(5x+15)}$ oe

Must be written in PF form, not just for correct *A* and *B*

M1: Area $R = \int \frac{15-3x}{(2x-4)(x+3)}\,dx = m\ln(2x-4)+n\ln(x+3)$

OR $\int \frac{15-3x}{(2x-4)(x+3)}\,dx = m\ln(x-2)+n\ln(x+3)$

Note that $\int \frac{l}{(x-2)}\,dx \to l\ln(kx-2k)$ and $\int \frac{m}{(x+3)}\,dx \to m\ln(nx+3n)$

A1ft: $= \int \frac{15-3x}{(2x-4)(x+3)}\,dx = 0.9\ln(2x-4)-2.4\ln(x+3)$ oe. FT on their *A* and *B*

B1: Deduces that the limits for the integral are 3 and 5. It cannot just be awarded from 5 being marked on Figure 4. So award for sight of $\int_3^5 \frac{15-3x}{(2x-4)(x+3)}(dx)$ or $[.............]_3^5$ having performed an integral which may be incorrect

dM1: Uses correct ln work seen at least once eg $\ln 6=\ln 2+\ln 3$, $\ln 8=3\ln 2$ or $m\ln 6k-m\ln 2k=m\ln 3$

This is an attempt to get either of the above ln's in terms of ln2 and/or ln3

It is dependent upon the correct limits and having achieved $m\ln(2x-4)+n\ln(x+3)$ oe

A1: $=3.3\ln 3-4.8\ln 2$ oe

2. The speed of a small jet aircraft was measured every 5 seconds, starting from the time it turned onto a runway, until the time when it left the ground.

The results are given in the table below with the time in seconds and the speed in $m\,s^{-1}$.

Time (s)	0	5	10	15	20	25
Speed ($m\,s^{-1}$)	2	5	10	18	28	42

Using all of this information,

(a) estimate the length of runway used by the jet to take off.

(3)

Given that the jet accelerated smoothly in these 25 seconds,

(b) explain whether your answer to part (a) is an underestimate or an overestimate of the length of runway used by the jet to take off.

(1)

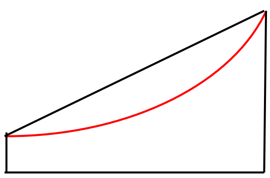
Question	Scheme	Marks	AOs
2	(data table below)		

Time (s)	0	5	10	15	20	25
Speed (m s^{-1})	2	5	10	18	28	42

Question	Scheme	Marks	AOs
(a)	Uses an allowable method to estimate the area under the curve. E.g.	M1	3.1a
	Way 1: an attempt at the trapezium rule (see below)		
	Way 2: $\{s=\}\left(\frac{2+42}{2}\right)(25)\ \{=550\}$		
	Way 3: $42=2+25(a) \Rightarrow a=1.6 \Rightarrow s=2(25)+(0.5)(1.6)(25)^2\{=550\}$		
	Way 4: $\{d=\}(2)(5)+5(5)+10(5)+18(5)+28(5)\ \{=63(5)=315\}$		
	Way 5: $\{d=\}\ 5(5)+10(5)+18(5)+28(5)+42(5)\ \{=103(5)=515\}$		
	Way 6: $\{d=\}\ \frac{315+515}{2}\ \{=415\}$		
	Way 7: $\{d=\}\left(\frac{2+5+10+18+28+42}{6}\right)(25)\ \{=437.5\}$		
	$\frac{1}{2}\times(5)\times[2+2(5+10+18+28)+42]$ or $\frac{1}{2}\times["315"+"515"]$	M1	1.1b
	$=415\ \{\text{m}\}$	A1	1.1b
		(3)	
(b) Alt 1	<u>**Uses a Way 1, Way 2, Way 3, Way 5, Way 6 or Way 7 method in (a)**</u>		
	Overestimate **and** a relevant explanation e.g. • {top of} trapezia lie above the curve • Area of trapezia > area under curve • An appropriate diagram which gives reference to the extra area • Curve is convex • $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}>0$ • Acceleration is {continually} increasing • The gradient of the curve is {continually} increasing • All the rectangles are above the curve (Way 5)	B1ft	2.4
		(1)	
(b) Alt 2	<u>**Uses a Way 4 method in (a)**</u>		
	Underestimate **and** a relevant explanation e.g. • All the rectangles are below the curve	B1ft	2.4
		(1)	
			(4 marks)

Notes for Question 2

(a)	
M1:	A low-level problem-solving mark for using an allowable method to estimate the area under the curve. E.g.
	Way 1: See scheme. Allow $\lambda(2+2(5+10+18+28)+42);\ \lambda>0$ for 1st M1
	Way 2: Uses $s=\left(\frac{u+v}{2}\right)t$ which is equivalent to finding the area of a large trapezium
	Way 3: Complete method using a uniform acceleration equation.
	Way 4: Sums rectangles lying below the curve. Condone a slip on one of the speeds.
	Way 5: Sums rectangles lying above the curve. Condone a slip on one of the speeds.
	Way 6: Average the result of Way 3 and Way 4. Equivalent to Way 1.
	Way 7: Applies (average speed)$\times$(time)

Notes for Question 2 Continued	
(a)	***continued***
M1:	Correct trapezium rule method with $h = 5$. Condone a slip on one of the speeds. The '2' and '42' should be in the correct place in the [......].
A1:	415
Note:	Units do not have to be stated
Note:	Give final A0 for giving a final answer with incorrect units. e.g. give final A0 for 415 km or 415ms^{-1}
Note:	Only the 1st M1 can only be scored for Way 2, Way 3, Way 4, Way 5 and Way 7 methods
Note:	Full marks in part (a) can only be scored by using a Way 1 or a Way 6 method.
Note:	Give M0 M0 A0 for $\{d=\}\ 2(5)+5(5)+10(5)+18(5)+28(5)+42(5)\ \{=105(5)=525\}$ (i.e. using too many rectangles)
Note	Condone M1 M0 A0 for $\left[\frac{(2+10)}{2}(10)+\frac{(10+18)}{2}(5)+\frac{(18+28)}{2}(5)+\frac{(28+42)}{2}(5)\right] = 395\text{ m}$
Note:	Give M1 M1 A1 for $5\left[\frac{(2+5)}{2}+\frac{(5+10)}{2}+\frac{(10+18)}{2}+\frac{(18+28)}{2}+\frac{(28+42)}{2}\right] = 415\text{ m}$
Note:	Give M1 M1 A1 for $\frac{5}{2}(2+42)+5(5+10+18+28) = 415\text{ m}$
Note:	**<u>Bracketing mistake:</u>**
	Unless the final calculated answer implies that the method has been applied correctly
	give M1 M0 A0 for $\frac{5}{2}(2)+2(5+10+18+28)+42\ \{=169\}$
	give M1 M0 A0 for $\frac{5}{2}(2+42)+2(5+10+18+28)\ \{=232\}$
Note:	Give M0 M0 A0 for a Simpson's Rule Method
(b)	**Alt 1**
B1ft:	**This mark depends on both an answer to part (a) being obtained and the first M in part (a)** See scheme
Note:	Allow the explanation "curve concaves upwards"
Note:	Do not allow explanations such as "curve is concave" or "curve concaves downwards"
Note:	Do not allow explanation "gradient of the curve is positive"
Note:	Do not allow explanations which refer to "friction" or "air resistance"
Note:	The diagram opposite is sufficient as an explanation. It must show the top of a trapezium lying above the curve. 
(b)	**Alt 2**
B1ft:	**This mark depends on both an answer to part (a) being obtained and the first M in part (a)** See scheme
Note:	Do not allow explanations which refer to "friction" or "air-resistance"

5.

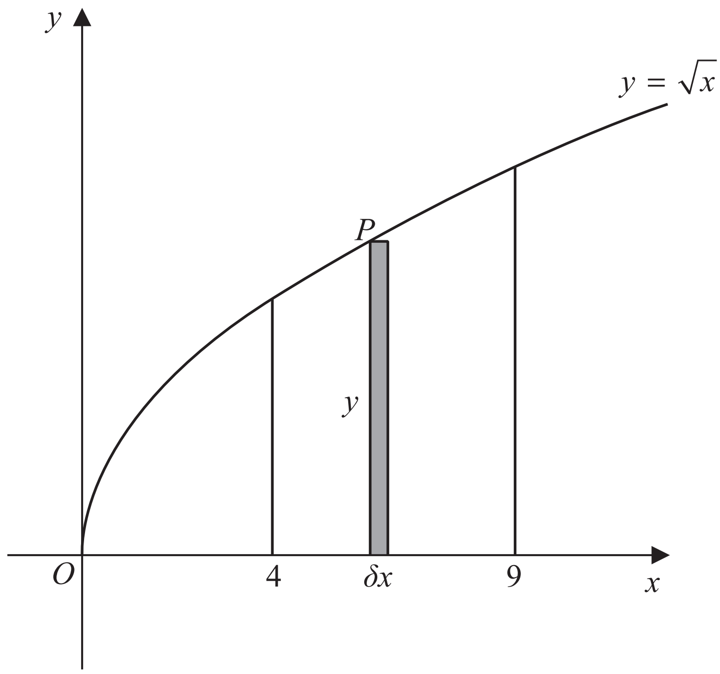


Figure 3

Figure 3 shows a sketch of the curve with equation $y = \sqrt{x}$

The point $P(x, y)$ lies on the curve.

The rectangle, shown shaded on Figure 3, has height y and width δx.

Calculate

$$\lim_{\delta x \to 0} \sum_{x=4}^{9} \sqrt{x}\, \delta x$$

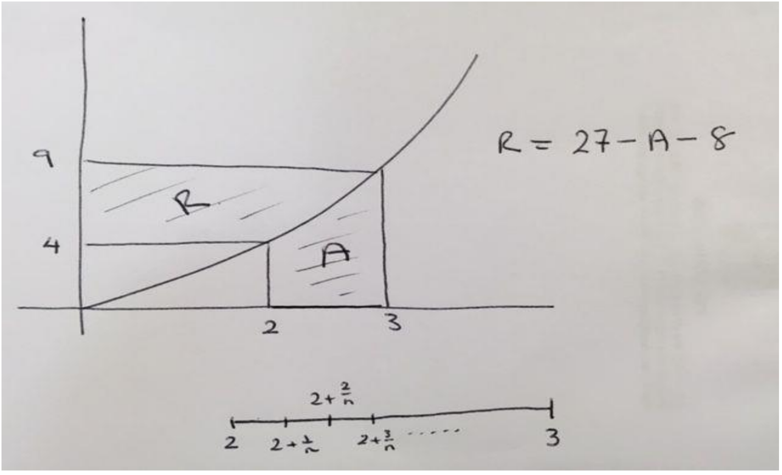
(3)

Question	Scheme	Marks	AOs
5	States $\left\{ \lim_{\delta x \to 0} \sum_{x=4}^{9} \sqrt{x}\ \delta x \text{ is} \right\}$ $\int_4^9 \sqrt{x}\,dx$	B1	1.2
	$= \left[\frac{2}{3} x^{\frac{3}{2}} \right]_4^9$	M1	1.1b
	$= \frac{2}{3} \times 9^{\frac{3}{2}} - \frac{2}{3} \times 4^{\frac{3}{2}} = \frac{54}{3} - \frac{16}{3}$		
	$= \frac{38}{3}$ or $12\frac{2}{3}$ or awrt 12.7	A1	1.1b
		(3)	
			(3 marks)

Notes for Question 5

B1:	States $\int_4^9 \sqrt{x}\,dx$ with or without the 'dx'
M1:	Integrates $\sqrt{x}$ to give $\lambda x^{\frac{3}{2}}$; $\lambda \neq 0$
A1:	See scheme
Note:	You can imply B1 for $\left[\lambda x^{\frac{3}{2}} \right]_4^9$ or for $\lambda \times 9^{\frac{3}{2}} - \lambda \times 4^{\frac{3}{2}}$
Note:	Give B0 for $\int_1^9 \sqrt{x}\,dx - \int_1^3 \sqrt{x}\,dx$ or for $\int_3^9 \sqrt{x}\,dx$ without reference to a correct $\int_4^9 \sqrt{x}\,dx$
Note:	Give B1 M1 A1 for no working leading to a correct $\frac{38}{3}$ or $12\frac{2}{3}$ or awrt 12.7
Note:	Give B1 M1 A1 for $\int_4^9 \sqrt{x}\,dx = \frac{38}{3}$ or $12\frac{2}{3}$ or awrt 12.7
Note:	Give B1 M1 A1 for $\left[\frac{2}{3} x^{\frac{3}{2}} + c \right]_4^9 = \frac{38}{3}$ or $12\frac{2}{3}$ or awrt 12.7
Note:	Give B1 M1 A1 for no working followed by an answer $\frac{38}{3}$ or $12\frac{2}{3}$ or awrt 12.7
Note:	Give M0 A0 for use of a trapezium rule method to give an answer of awrt 12.7, but allow B1 if $\int_4^9 \sqrt{x}\,dx$ is seen in a trapezium rule method
Note:	Otherwise, give B0 M0 A0 for using the trapezium rule to give an answer of awrt 12.7

Notes for Question 5 Continued	
Alt	The following method is correct:



$$\text{Area }(A) = \lim_{n\to\infty}\sum_{i=1}^{n}(x_i - x_{i-1})\mathrm{f}(x_i) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(2+\frac{i}{n}\right)^2$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}4 + \frac{1}{n}\sum_{i=1}^{n}\left(\frac{4i}{n}\right) + \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i^2}{n^2}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}4 + \frac{4}{n^2}\sum_{i=1}^{n}i + \frac{1}{n^3}\sum_{i=1}^{n}i^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{4n}{n} + \frac{4}{n^2}\left(\frac{1}{2}n(n+1)\right) + \frac{1}{n^3}\left(\frac{1}{6}n(n+1)(2n+1)\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{4}{n} + \frac{4n^2+4n}{2n^2} + \frac{2n^3+3n^2+n}{6n^3}\right]$$

$$= \lim_{n\to\infty}\left[4 + 2 + \frac{2}{n} + \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right]$$

$$= 4 + 2 + \frac{1}{3} = \frac{19}{3}$$

So, $\lim_{\delta x\to 0}\sum_{x=4}^{9}\sqrt{x}\ \delta x = \text{Area}(R) = (3\times 9) - (2\times 4) - \frac{19}{3}$

$$= \frac{38}{3} \quad \text{or} \quad 12\frac{2}{3} \quad \text{or} \quad \text{awrt } 12.7$$

14. (a) Use the substitution $u = 4 - \sqrt{h}$ to show that

$$\int \frac{\mathrm{d}h}{4 - \sqrt{h}} = -8\ln\left|4 - \sqrt{h}\right| - 2\sqrt{h} + k$$

where k is a constant

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.25}\left(4 - \sqrt{h}\right)}{20}$$

where h is the height in metres and t is the time, measured in years, after the tree is planted.

(b) Find, according to the model, the range in heights of trees in this species.

(2)

One of these trees is one metre high when it is first planted.

According to the model,

(c) calculate the time this tree would take to reach a height of 12 metres, giving your answer to 3 significant figures.

(7)

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Question	Scheme	Marks	AOs
14 (a)	$\{u = 4 - \sqrt{h} \Rightarrow\}\ \dfrac{du}{dh} = -\dfrac{1}{2}h^{-\frac{1}{2}}$ or $\dfrac{dh}{du} = -2(4-u)$ or $\dfrac{dh}{du} = -2\sqrt{h}$	B1	1.1b
	$\left\{\displaystyle\int \frac{dh}{4-\sqrt{h}} = \right\}\ \displaystyle\int \frac{-2(4-u)}{u}\,du$	M1	2.1
	$= \displaystyle\int \left(-\frac{8}{u} + 2\right) du$	M1	1.1b
	$= -8\ln u + 2u\ \{+c\}$	M1	1.1b
		A1	1.1b
	$= -8\ln\|4-\sqrt{h}\| + 2(4-\sqrt{h}) + c\ =\ -8\ln\|4-\sqrt{h}\| - 2\sqrt{h}\ + k$ *	A1*	2.1
		(6)	
(b)	$\left\{\dfrac{dh}{dt} = \dfrac{t^{0.25}(4-\sqrt{h})}{20} = 0 \Rightarrow\right\}\ 4 - \sqrt{h} = 0$	M1	3.4
	Deduces any of $0 < h < 16,\ 0 \le h < 16,\ 0 < h \le 16,\ 0 \le h \le 16,$ $h < 16,\ h \le 16$ or all values up to 16	A1	2.2a
		(2)	
(c) Way 1	$\displaystyle\int \frac{1}{(4-\sqrt{h})}\,dh\ =\ \int \frac{1}{20}t^{0.25}\,dt$	B1	1.1b
	$-8\ln\|4-\sqrt{h}\| - 2\sqrt{h}\ = \dfrac{1}{25}t^{1.25}\ \ \{+c\}$	M1	1.1b
		A1	1.1b
	$\{t = 0,\ h = 1 \Rightarrow\}\ -8\ln(4-1) - 2\sqrt{(1)}\ = \dfrac{1}{25}(0)^{1.25}\ + c$	M1	3.4
	$\Rightarrow c = -8\ln(3) - 2 \Rightarrow\ -8\ln\|4-\sqrt{h}\| - 2\sqrt{h}\ = \dfrac{1}{25}t^{1.25} - 8\ln(3) - 2$ $\{h = 12 \Rightarrow\}\ -8\ln\|4-\sqrt{12}\| - 2\sqrt{12}\ = \dfrac{1}{25}t^{1.25} - 8\ln(3) - 2$	dM1	3.1a
	$t^{1.25} = 221.2795202... \Rightarrow\ t = \sqrt[1.25]{221.2795...}$ or $t = (221.2795...)^{0.8}$	M1	1.1b
	$t = 75.154... \Rightarrow t =\ 75.2$ (years) (3 sf) or awrt 75.2 (years)	A1	1.1b
	Note: You can recover work for part (c) in part (b)	**(7)**	
(c) Way 2	$\displaystyle\int_1^{12} \frac{20}{(4-\sqrt{h})}\,dh\ =\ \int_0^T t^{0.25}\,dt$	B1	1.1b
	$\left[20(-8\ln\|4-\sqrt{h}\| - 2\sqrt{h})\right]_1^{12}\ = \left[\dfrac{4}{5}t^{1.25}\right]_0^T$	M1	1.1b
		A1	1.1b
	$20(-8\ln(4-\sqrt{12}) - 2\sqrt{12}) - 20(-8\ln(4-1) - 2\sqrt{1}) = \dfrac{4}{5}T^{1.25} - 0$	M1	3.4
		dM1	3.1a
	$T^{1.25} = 221.2795202... \Rightarrow\ T = \sqrt[1.25]{221.2795...}$ or $T = (221.2795...)^{0.8}$	M1	1.1b
	$T = 75.154... \Rightarrow T =\ 75.2$ (years) (3 sf) or awrt 75.2 (years)	A1	1.1b
	Note: You can recover work for part (c) in part (b)	**(7)**	
			(15 marks)

Notes for Question 14	
(a)	
B1:	See scheme. Allow $\mathrm{d}u = -\frac{1}{2}h^{-\frac{1}{2}}\mathrm{d}h$, $\mathrm{d}h = -2(4-u)\mathrm{d}u$, $\mathrm{d}h = -2\sqrt{h}\,\mathrm{d}u$ o.e.
M1: **Note:**	Complete method for applying $u = 4-\sqrt{h}$ to $\int \frac{\mathrm{d}h}{4-\sqrt{h}}$ to give an expression of the form $\int \frac{k(4-u)}{u}\mathrm{d}u$; $k \neq 0$ Condone the omission of an integral sign and/or $\mathrm{d}u$
M1:	Proceeds to obtain an integral of the form $\int\left(\frac{A}{u}+B\right)\{\mathrm{d}u\}$; $A, B \neq 0$
M1:	$\int\left(\frac{A}{u}+B\right)\{\mathrm{d}u\} \to D\ln u + Eu$; $A, B, D, E \neq 0$; with or without a constant of integration
A1:	$\int\left(-\frac{8}{u}+2\right)\{\mathrm{d}u\} \to -8\ln u + 2u$; with or without a constant of integration
A1*:	**dependent on all previous marks** Substitutes $u = 4-\sqrt{h}$ into their integrated result and completes the proof by obtaining the printed result $-8\ln\left\|4-\sqrt{h}\right\| - 2\sqrt{h} + k$. Condone the use of brackets instead of the modulus sign.
Note:	They must combine $2(4)$ and their $+c$ correctly to give $+k$
Note:	Going from $-8\ln\left\|4-\sqrt{h}\right\| + 2(4-\sqrt{h}) + c$ to $-8\ln\left\|4-\sqrt{h}\right\| - 2\sqrt{h} + k$, with no intermediate working or with no incorrect working is required for the final A1* mark.
Note:	Allow A1* for correctly reaching $-8\ln\left\|4-\sqrt{h}\right\| - 2\sqrt{h} + c + 8$ and stating $k = c+8$
Note:	Allow A1* for correctly reaching $-8\ln\left\|4-\sqrt{h}\right\| + 2(4-\sqrt{h}) + k = -8\ln\left\|4-\sqrt{h}\right\| - 2\sqrt{h} + k$
	Alternative (integration by parts) method for the 2nd M, 3rd M and 1st A mark
	$\left\{\int \frac{-2(4-u)}{u}\mathrm{d}u = \int \frac{2u-8}{u}\mathrm{d}u\right\} = (2u-8)\ln u - \int 2\ln u\,\mathrm{d}u = (2u-8)\ln u - 2(u\ln u - u)\ \{+c\}$
2nd M1:	Proceeds to obtain an integral of the form $(Au+B)\ln u - \int A\ln u\{\mathrm{d}u\}$; $A, B \neq 0$
3rd M1:	Integrates to give $D\ln u + Eu$; $D, E \neq 0$; which can be simplified or un-simplified with or without a constant of integration.
Note:	Give 3rd M1 for $(2u-8)\ln u - 2(u\ln u - u)$ because it is an un-simplified form of $D\ln u + Eu$
1st A1:	Integrates to give $(2u-8)\ln u - 2(u\ln u - u)$ or $-8\ln u + 2u$ o.e. with or without a constant of integration.
(b)	
M1: **Note:**	Uses the context of the model and has an understanding that the tree keeps growing until $\frac{\mathrm{d}h}{\mathrm{d}t} = 0 \Rightarrow 4-\sqrt{h} = 0$. Alternatively, they can write $\frac{\mathrm{d}h}{\mathrm{d}t} > 0 \Rightarrow 4-\sqrt{h} > 0$ Accept $h = 16$ or 16 used in their inequality statement for this mark.
A1:	See scheme
Note:	A correct answer can be given M1 A1 from any working.

Notes for Question 14	
(c)	**Way 1**
B1:	Separates the variables correctly. $\mathrm{d}h$ and $\mathrm{d}t$ should not be in the wrong positions, although this mark can be implied by later working. Condone absence of integral signs.
M1:	Integrates $t^{0.25}$ to give $\lambda t^{1.25}$; $\lambda \neq 0$
A1:	Correct integration. E.g. $-8\ln\left\|4-\sqrt{h}\right\|-2\sqrt{h} = \frac{1}{25}t^{1.25}$ or $20(-8\ln\left\|4-\sqrt{h}\right\|-2\sqrt{h}) = \frac{4}{5}t^{1.25}$ $-8\ln\left\|4-\sqrt{h}\right\|+2(4-\sqrt{h}) = \frac{1}{25}t^{1.25}$ or $20(-8\ln\left\|4-\sqrt{h}\right\|+2(4-\sqrt{h})) = \frac{4}{5}t^{1.25}$ with or without a constant of integration, e.g. *k*, *c* or *A*
Note:	There is no requirement for modulus signs.
M1:	Some evidence of ***applying*** both $t=0$ and $h=1$ to their model (which can be a changed equation) which contains a constant of integration, e.g. *k*, *c* or *A*
dM1:	**dependent on the previous M mark** Complete process of finding their constant of integration, followed by applying $h=12$ and their constant of integration to their changed equation
M1:	Rearranges their equation to make $t^{\text{their } 1.25} = ...$ followed by a correct method to give $t = ...$; $t > 0$
Note:	$t^{\text{their } 1.25} = ...$ can be negative, but their '$t = ...$' must be positive
Note:	"their 1.25" cannot be 0 or 1 for this mark
Note:	Do not give this mark if $t^{\text{their } 1.25} = ...$ (usually $t^{0.25} = ...$) is a result of substituting $t=12$ (or $t=11$) into the given $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.25}(4-\sqrt{h})}{20}$. **Note:** They will usually write $\frac{\mathrm{d}h}{\mathrm{d}t}$ as either 12 or 11.
A1:	awrt 75.2
(c)	**Way 2**
B1:	Separates the variables correctly. $\mathrm{d}h$ and $\mathrm{d}t$ should not be in the wrong positions, although this mark can be implied by later working.
Note:	Integral signs and limits are not required for this mark.
M1:	Same as Way 1 (ignore limits)
A1:	Same as Way 1 (ignore limits)
M1:	Applies limits of 1 and 12 to their model (i.e. to their changed expression in *h*) and subtracts
dM1	**dependent on the previous M mark** Complete process of applying limits of 1 and 12 and 0 and *T* (or '*t*') appropriately to their changed equation
M1:	Same as Way 1
A1:	Same as Way 1

10. (a) Use the substitution $x = u^2 + 1$ to show that

$$\int_5^{10} \frac{3\,\mathrm{d}x}{(x-1)\left(3+2\sqrt{x-1}\right)} = \int_p^q \frac{6\,\mathrm{d}u}{u(3+2u)}$$

where p and q are positive constants to be found.

(4)

(b) Hence, using algebraic integration, show that

$$\int_5^{10} \frac{3\,\mathrm{d}x}{(x-1)\left(3+2\sqrt{x-1}\right)} = \ln a$$

where a is a rational constant to be found.

(6)

Question	Scheme	Marks	AOs
10 (a)	$x=u^2+1 \Rightarrow \mathrm{d}x = 2u\mathrm{d}u$ oe	B1	1.1b
	Full substitution $\int \frac{3\mathrm{d}x}{(x-1)\left(3+2\sqrt{x-1}\right)} = \int \frac{3\times 2u\ \mathrm{d}u}{\left(u^2+1-1\right)(3+2u)}$	M1	1.1b
	Finds correct limits e.g. $p=2, q=3$	B1	1.1b
	$= \int \frac{3\times 2\not{u}\ \mathrm{d}u}{u^{\not{2}}(3+2u)} = \int \frac{6\ \mathrm{d}u}{u(3+2u)}$ *	A1*	2.1
		(4)	
(b)	$\frac{6}{u(3+2u)} = \frac{A}{u} + \frac{B}{3+2u} \Rightarrow A = ..., B = ...$	M1	1.1b
	Correct PF. $\frac{6}{u(3+2u)} = \frac{2}{u} - \frac{4}{3+2u}$	A1	1.1b
	$\int \frac{6\ \mathrm{d}u}{u(3+2u)} = 2\ln u - 2\ln(3+2u) \quad (+c)$	dM1 A1ft	3.1a 1.1b
	Uses limits $u = "3", u = "2"$ with some correct ln work leading to $k\ln b$ E.g. $(2\ln 3 - 2\ln 9) - (2\ln 2 - 2\ln 7) = 2\ln\frac{7}{6}$	M1	1.1b
	$\ln\frac{49}{36}$	A1	2.1
		(6)	
			(10 marks)
Notes: Mark (a) and (b) together as one complete question			

(a)

B1: $\mathrm{d}x = 2u\mathrm{d}u$ or exact equivalent. E.g. $\frac{\mathrm{d}x}{\mathrm{d}u} = 2u$, $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2}(x-1)^{-\frac{1}{2}}$

M1: Attempts a full substitution of $x = u^2+1$, including $\mathrm{d}x \to ...u\mathrm{d}u$ to form an integrand in terms of u. Condone slips but there should be an attempt to use the correct substitution on the denominator.

B1: Finds correct limits either states $p=2, q=3$ or sight of embedded values as limits to the integral

A1*: Clear reasoning including one fully correct intermediate line, including the integral signs, leading to the given expression ignoring limits. So B1, M1, B0, A1 is possible if the limits are incorrect, omitted or left as 5 and 10.

(b)

M1: Uses correct form of PF leading to values of A and B.

A1: Correct PF $\frac{6}{u(3+2u)} = \frac{2}{u} - \frac{4}{3+2u}$ (Not scored for just the correct values of A and B)

dM1: This is an overall problem solving mark. It is for using the correct PF form and integrating using lns. Look for $P\ln u + Q\ln(3+2u)$

A1ft: Correct integration for their $\frac{A}{u} + \frac{B}{3+2u} \to A\ln u + \frac{B}{2}\ln(3+2u)$ with or without modulus signs

M1: Uses their 2 and 3 as limits, with at least one correct application of the addition law or subtraction law leading to the form $k\ln b$ or $\ln a$. PF's must have been attempted. Condone bracketing slips. Alternatively changing the u's back to x's and use limits of 5 and 10.

A1: Proceeds to $\ln\frac{49}{36}$. Answers without working please send to review.

14. A large spherical balloon is deflating.

At time t seconds the balloon has radius r cm and volume V cm^3

The volume of the balloon is modelled as decreasing at a constant rate.

(a) Using this model, show that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{k}{r^2}$$

where k is a positive constant.

(3)

Given that

- the initial radius of the balloon is 40 cm
- after 5 seconds the radius of the balloon is 20 cm
- the volume of the balloon continues to decrease at a constant rate until the balloon is empty

(b) solve the differential equation to find a complete equation linking r and t.

(5)

(c) Find the limitation on the values of t for which the equation in part (b) is valid.

(2)

Question	Scheme	Marks	AOs
14 (a)	Uses the model to state $\frac{dV}{dt}=-c$ (for positive constant c)	B1	3.1b
	Uses $\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$ with their $\frac{dV}{dt}=-c$ and $\frac{dV}{dr}=4\pi r^2$	M1	2.1
	$-c=4\pi r^2\times\frac{dr}{dt}\Rightarrow\frac{dr}{dt}=-\frac{c}{4\pi r^2}=-\frac{k}{r^2}$ *	A1*	2.2a
		(3)	
(b)	$\frac{dr}{dt}=-\frac{k}{r^2}\Rightarrow\int r^2 dr=\int -k dt$ and integrates with one side "correct"	M1	2.1
	$\frac{r^3}{3}=-kt(+\alpha)$	A1	1.1b
	Uses $t=0, r=40\Rightarrow\alpha=...$ $\quad\alpha=\frac{64000}{3}$	M1	1.1b
	Uses $t=5, r=20$ & $\alpha=...\Rightarrow k=...$	M1	3.4
	$r^3=64000-11200t$ or exact equivalent	A1	3.3
		(5)	
(c)	Uses the equation of their model and proceeds to a limiting value for t E.g. $"64000-11200t"\ ...\ 0\Rightarrow t...$	M1	3.4
	For times up to and including $\frac{40}{7}$ seconds	A1ft	3.5b
		(2)	
		(10 marks)	

Notes:

(a)

B1: Uses the model to state $\frac{\mathrm{d}V}{\mathrm{d}t} = -c$ (for positive constant c).

Any "letter" is acceptable here including k.

Note that $\frac{\mathrm{d}V}{\mathrm{d}t} = c$ is B0 unless they state that c is a negative constant.

M1: For an attempt to use $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$ with their $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$

Allow for an attempt to use $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$ with their $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $\frac{\mathrm{d}V}{\mathrm{d}r} = \lambda r^2$ (Any constant is fine)

There is no requirement to use the correct formula for the volume of a sphere for this mark.

A1*: Proceeds to the given answer with an intermediate line equivalent to $\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{c}{4\pi r^2}$

If candidate started with $\frac{\mathrm{d}V}{\mathrm{d}t} = -k$ they must provide a minimal explanation how

$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{k}{4\pi r^2} \rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{k}{r^2}$. E.g $\frac{1}{4\pi}$ is a constant so replace $\frac{k}{4\pi}$ with k

It is not necessary to use the full formula for the volume of a sphere, eg allow $V = \kappa r^3$ but if it has been quoted it must be correct. So using $V = 4r^3$ can potentially score 2 of the 3 marks.

(b)

M1: For the key step of separating the variables correctly AND integrating one side with at least one index correct. The integral signs do not need to be seen.

A1: Correct integration E.g. $\frac{r^3}{3} = -kt(+\alpha)$ or equivalent. The $+\alpha$ is not required for this mark.

This may be awarded if k has been given a value.

M1: Uses the initial conditions to find a value for the constant of integration α

If a constant of integration is not present, or k has been given a pre defined value, then only the first two marks can be awarded in part (b)

The mark may be awarded if the equation has been adapted incorrectly. E.g. each term cube rooted.

M1: Uses the second set of conditions with their value of α to find k

This may be awarded if the equation has been adapted incorrectly. E.g. each term cube rooted.

A1: Obtains any correct equation for the model.

E.g. $r^3 = 64000 - 11200t$ or exact equivalent such as $\frac{r^3}{3} = \frac{64000}{3} - \frac{11200}{3}t$.

ISW after sight of a correct answer. Condone recurring decimals e.g. $21333.\dot{3}$ for $\frac{64000}{3}$

Do not award if **only the** rounded/truncated decimal equivalents to say $\frac{64000}{3}$ is used.

(c)

M1: Recognises that the model is only valid when $r \geqslant 0$ **and uses this to find *t***. Condone $r > 0$

Award for an attempt to find the value of t when r =0. See scheme.

It must be from an equation of the form $ar^n = b - ct, \quad a,b,c > 0$ which give + ve values of t.

A1ft: Allow valid for times up to (and including) $\frac{40}{7}$ seconds, 5.71 seconds. Allow $t < \frac{40}{7}$ or $t \leqslant \frac{40}{7}$

There is no requirement for the left hand side of the inequality, 0

States invalid for times greater than $\frac{40}{7}$ seconds, 5.71 seconds.

Follow through on their equation so allow $t <$ their "$\frac{64000}{11200}$" as long as this value is greater than 5

(t = 5 is one of the values in the question)

1 The table below shows corresponding values of x and y for $y = \sqrt{\frac{x}{1+x}}$

The values of y are given to 4 significant figures.

x	0.5	1	1.5	2	2.5
y	0.5774	0.7071	0.7746	0.8165	0.8452

(a) Use the trapezium rule, with all the values of y in the table, to find an estimate for

$$\int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}}\, dx$$

giving your answer to 3 significant figures.

(3)

(b) Using your answer to part (a), deduce an estimate for $\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\, dx$

(1)

Given that

$$\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\, dx = 4.535 \text{ to 4 significant figures}$$

(c) comment on the accuracy of your answer to part (b).

(1)

Question	Scheme	Marks	AOs
1(a)	$h = 0.5$	B1	1.1a
	$A \approx \frac{0.5}{2}\left\{0.5774 + 0.8452 + 2\left(0.7071 + 0.7746 + 0.8165\right)\right\}$	M1	1.1b
	= awrt 1.50	A1	1.1b
	For reference: **The integration on a calculator gives 1.511549071** **The full accuracy for *y* values gives 1.504726147** **The accuracy from the table gives 1.50475**		
		(3)	

(b)	3 × their (a)
	If (a) is correct, allow awrt 4.50 or awrt 4.51 even with no working.
	Only allow 4.5 if (a) is correct and working is shown e.g. 3 × 1.5
	
	If (a) is incorrect allow 3 × their (a) given to at least 3sf but do not be too concerned about the accuracy (as they may use rounded or rounded value from (a))
		B1ft	2.2a

	For reference the integration on a calculator gives 4.534647213		
		(1)	
(c)	**<u>This mark depends on the B1 having been awarded in part (b) with awrt 4.5</u>** Look for a sensible comment. Some examples: • The answer is accurate to 2 sf or one decimal place • Answer to (b) is accurate as 4.535 ≈ 4.50 • Very accurate as 4.535 to 2 sf is 4.5 • $4.51425 < 4.535$ so my answer is underestimate but not too far off • It is an underestimate but quite close • It is a very good estimate • High accuracy • (Quite) accurate • It is less than 1% out • 4.535 – 4.5 = 0.035 so not far out But **not** just "it is an underestimate" or Calculates percentage error correctly using awrt 4.50 or awrt 4.51 or 4.5 (No comment is necessary in these cases although one may be given) Examples: $\left\|\frac{4.535-4.50}{4.535}\right\|\times 100 = 0.77\%$ or $\left\|\frac{4.535-4.51}{4.535}\right\|\times 100 = 0.55\%$ or $\left\|\frac{4.535-4.51425}{4.535}\right\|\times 100 = 0.46\%$ or $\left\|\frac{4.50}{4.535}\right\|\times 100 = 99\%$ In these cases don't be too concerned about accuracy e.g. allow 1sf. **This mark should be withheld if there are any contradictory statements**	B1	3.2b
		(1)	
			(5 marks)

Notes:

(a)

B1: States or uses $h = 0.5$. May be implied by $\frac{1}{4}\times\{\ldots$ below.

M1: Correct attempt at the trapezium rule.

Look for $\frac{1}{2}h\times\{0.5774+0.8452+2(0.7071+0.7746+0.8165)\}$ condoning slips on the terms but must use all y values with no repeats.

There must be a clear attempt at $\frac{1}{2}h\times$(first y + last y + $2\times$"sum of the rest")

Give M0 for $\frac{1}{2}\times\frac{1}{2}0.5774+0.8452+2(0.7071+0.7746+0.8165)$ unless the missing brackets are implied.

NB this incorrect method gives 5.85…

May be awarded for separate trapezia e.g.

$$\frac{1}{4}(0.5774+0.7071)+\frac{1}{4}(0.7071+0.7746)+\frac{1}{4}(0.7746+0.8165)+\frac{1}{4}(0.8165+0.8452)$$

May be awarded for using the function e.g. $\frac{1}{2}h\times\left\{\sqrt{\frac{0.5}{1+0.5}}+\sqrt{\frac{2.5}{1+2.5}}+2\left(\sqrt{\frac{1}{1+1}}+\sqrt{\frac{1.5}{1+1.5}}+\sqrt{\frac{2}{1+2}}\right)\right\}$

A1: Awrt 1.50 (Apply isw if necessary)

Correct answers with no working – send to review

(b)

B1ft: See main scheme. Must be considering $3\times$ (a) and not e.g. attempting trapezium rule again.

(c)

B1: See scheme

6. (a) Given that

$$\frac{x^2 + 8x - 3}{x + 2} \equiv Ax + B + \frac{C}{x + 2} \qquad x \in \mathbb{R} \;\; x \neq -2$$

find the values of the constants A, B and C

(3)

(b) Hence, using algebraic integration, find the exact value of

$$\int_0^6 \frac{x^2 + 8x - 3}{x + 2}\, dx$$

giving your answer in the form $a + b\ln 2$ where a and b are integers to be found.

(4)

Question	Scheme	Marks	AOs
6(a)	$x^2+8x-3=(Ax+B)(x+2)+C$ or $Ax(x+2)+B(x+2)+C$ $\Rightarrow A=\ldots, B=\ldots, C=\ldots$ **or** $x+2\,\overline{)\,x^2+8x-3}$ quotient $x+6$ $\underline{x^2+2x}$ $6x-3$ $\underline{6x+12}$ -15	M1	1.1b
	Two of $A=1, B=6, C=-15$	A1	1.1b
	All three of $A=1, B=6, C=-15$	A1	1.1b
		(3)	
6(b)	$\int \frac{x^2+8x-3}{x+2}\,\mathrm{d}x=\int x+6-\frac{15}{x+2}\,\mathrm{d}x=\ldots-15\ln(x+2)$	M1	1.1b
	$=\frac{1}{2}x^2+6x-15\ln(x+2)\quad(+c)$	A1ft	1.1b
	$\int_0^6 \frac{x^2+8x-3}{x+2}\,\mathrm{d}x=\left[\frac{1}{2}x^2+6x-15\ln(x+2)\right]_0^6$ $=(18+36-15\ln8)-(0+0-15\ln2)$ $=18+36-(15-45)\ln2$ or e.g. $18+36+15\ln\left(\frac{2}{8}\right)$	M1	2.1
	$=54-30\ln2$	A1	1.1b
		(4)	
			(7 marks)

Notes:

(a)

M1: Multiplies by $(x + 2)$ and attempts to find values for A, B and C e.g. by comparing coefficients or substituting values for x. If the method is unclear, at least 2 terms must be correct on rhs.

Or attempts to divide x^2+8x-3 by $x + 2$ and obtains a linear quotient and a constant remainder.

This mark may be implied by 2 correct values for A, B or C

A1: Two of $A = 1$, $B = 6$, $C = -15$. But note that **just** performing the division correctly is insufficient and they must clearly identify their A, B, C to score any accuracy marks.

A1: All three of $A = 1$, $B = 6$, $C = -15$

This is implied by stating $\frac{x^2+8x-3}{x+2}=x+6-\frac{15}{x+2}$ or within the integral in (b)

(b)

M1: Integrates an expression of the form $\frac{C}{x+2}$ to obtain $k\ln(x + 2)$.

Condone the omission of brackets around the "$x + 2$"

A1ft: Correct integration ft on their $Ax+B+\frac{C}{x+2}$, $(A,B,C\neq 0)$ The brackets should be present around the "$x + 2$" unless they are implied by subsequent work.

M1: Substitutes both limits 0 and 6 into an expression that contains an x or x^2 term or both and a ln term and subtracts either way round **WITH** fully correct log work to combine two log terms (but allow sign errors when removing brackets) leading to an answer of the form $a + b\ln c$ (a, b and c not necessarily integers)

e.g. if they expand to get $-15\ln 8 - 15\ln 2$ followed by $-15\ln 16$ and reach $a + b\ln c$ then allow the M mark

A1: $54 - 30\ln 2$ (Apply isw once a correct answer is seen)

12.

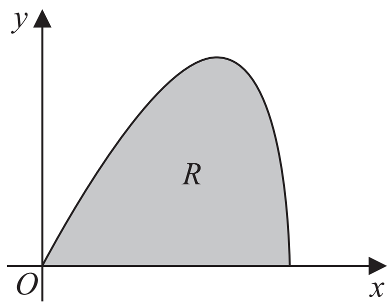

Figure 3

The curve shown in Figure 3 has parametric equations

$$x = 6\sin t \qquad y = 5\sin 2t \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

The region R, shown shaded in Figure 3, is bounded by the curve and the x-axis.

(a) (i) Show that the area of R is given by $\displaystyle\int_0^{\frac{\pi}{2}} 60\sin t\cos^2 t\,\mathrm{d}t$

(3)

(ii) Hence show, by algebraic integration, that the area of R is exactly 20

(3)

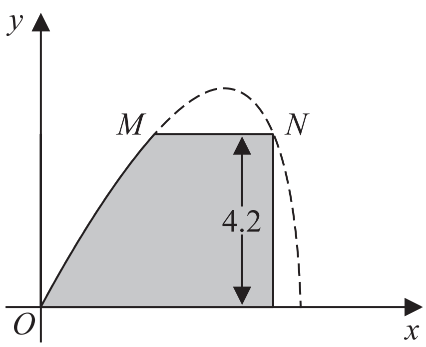

Figure 4

Part of the curve is used to model the profile of a small dam, shown shaded in Figure 4.
Using the model and given that

- x and y are in metres
- the vertical wall of the dam is 4.2 metres high
- there is a horizontal walkway of width MN along the top of the dam

(b) calculate the width of the walkway.

(5)

Question	Scheme	Marks	AOs
12(a)(i)	$y\times\frac{\mathrm{d}x}{\mathrm{d}t}=5\sin 2t\times 6\cos t$ or $5\times 2\sin t\cos t\times 6\cos t$	M1	1.2
	(Area =) $\int 5\sin 2t\times 6\cos t\,\mathrm{d}t=\int 5\times 2\sin t\cos t\times 6\cos t\,\mathrm{d}t$ or $\int 5\sin 2t\times 6\cos t\,\mathrm{d}t=\int 60\sin t\cos^2 t\,\mathrm{d}t$	dM1	1.1b
	(Area =) $\int_0^{\frac{\pi}{2}} 60\sin t\cos^2 t\,\mathrm{d}t$ *	A1*	2.1*
		(3)	
(a)(ii)	$\int 60\sin t\cos^2 t\,\mathrm{d}t=-20\cos^3 t$	M1 A1	1.1b 1.1b
	Area $=\left[-20\cos^3 t\right]_0^{\frac{\pi}{2}}=0-(-20)=20$ *	A1*	2.1
		(3)	
(b)	$5\sin 2t=4.2\Rightarrow \sin 2t=\frac{4.2}{5}$	M1	3.4
	$t=0.4986...,1.072...$	A1	1.1b
	Attempts to finds the x values at both t values	dM1	3.4
	$t=0.4986...\Rightarrow x=2.869...$ $t=1.072\Rightarrow x=5.269...$	A1	1.1b
	Width of path = 2.40 metres	A1	3.2a
		(5)	
			(11 marks)

Notes:

(a)(i)

M1: Attempts to multiply y by $\frac{\mathrm{d}x}{\mathrm{d}t}$ to obtain $A\sin 2t\cos t$ but may apply $\sin 2t = 2\sin t\cos t$ here

dM1: Attempts to use $\sin 2t = 2\sin t\cos t$ within an integral which may be implied by

e.g. $A\int \sin 2t \times \cos t \, \mathrm{d}t = \int k\sin t\cos^2 t \, \mathrm{d}t$

A1*: Fully correct work leading to the given answer.
This must include $\sin 2t = 2\sin t\cos t$ or e.g. $5\sin 2t = 10\sin t\cos t$ seen explicitly in their proof and a correct intermediate line that includes an integral sign and the "dt"
Allow the limits to just "appear" in the final answer e.g. working need not be shown for the limits.

(a)(ii)

M1: Obtains $\int 60\sin t\cos^2 t \, \mathrm{d}t = k\cos^3 t$. This may be attempted via a substitution of $u = \cos t$ to obtain

$\int 60\sin t\cos^2 t \, \mathrm{d}t = ku^3$

A1: Correct integration $-20\cos^3 t$ or equivalent e.g. $-20u^3$

A1*: Rigorous proof with all aspects correct including the correct limits and the $0-(-20)$ and

not just: $-20\cos^3\frac{\pi}{2} - \left(-20\cos^3 0\right) = 20$

(b)

M1: Uses the given model and attempts to find value(s) of t when $\sin 2t = \frac{4.2}{5}$. Look for $2t = \sin^{-1}\frac{4.2}{5} \Rightarrow t = ...$

A1: At least one correct value for t, correct to 2 dp. FYI $t = 0.4986...,1.072...$ or in degrees $t = 28.57…, 61.42…$

dM1: Attempts to find **TWO** distinct values of x when $\sin 2t = \frac{4.2}{5}$. Condone poor trig work and allow this mark if 2 values of x are attempted from 2 values of t.

A1: Both values correct to 2 dp. NB $x = 2.869…, 5.269…$

Or may take Cartesian approach

$$5\sin 2t = 4.2 \Rightarrow 10\sin t\cos t = 4.2 \Rightarrow 10\frac{x}{6}\sqrt{1-\frac{x^2}{36}} = 4.2 \Rightarrow x^4 - 36x^2 + 228.6144 = 0 \Rightarrow x = 2.869...,5.269...$$

M1: For converting to Cartesian form A1: Correct quartic M1: Solves quartic A1: Correct values

A1: 2.40 metres or 240 cm
Allow awrt 2.40 m or allow 2.4m (not awrt 2.4 m) and allow awrt 240 cm. **Units are required**.

11.

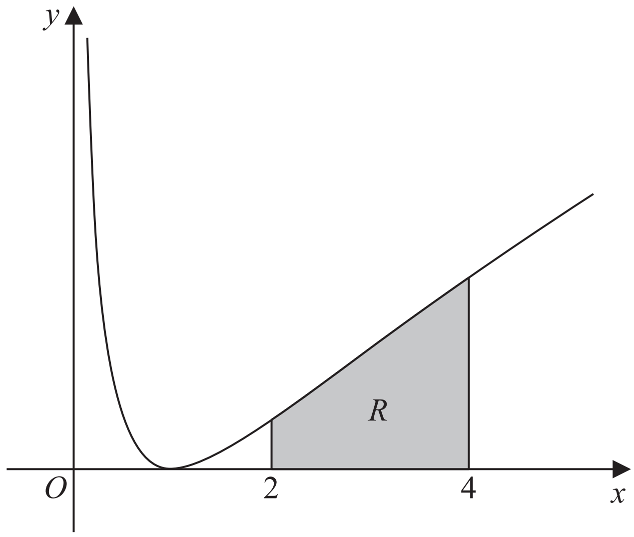

Figure 2

Figure 2 shows a sketch of part of the curve with equation

$$y = (\ln x)^2 \qquad x > 0$$

The finite region R, shown shaded in Figure 2, is bounded by the curve, the line with equation $x = 2$, the x-axis and the line with equation $x = 4$

The table below shows corresponding values of x and y, with the values of y given to 4 decimal places.

x	2	2.5	3	3.5	4
y	0.4805	0.8396	1.2069	1.5694	1.9218

(a) Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of R, giving your answer to 3 significant figures.

(3)

(b) Use algebraic integration to find the exact area of R, giving your answer in the form

$$y = a(\ln 2)^2 + b\ln 2 + c$$

where a, b and c are integers to be found.

(5)

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P 6 8 7 3 1 A 0 3 4 5 2

Question	Scheme	Marks	AOs
11(a)	$h=0.5$	B1	1.1b
	$A\approx\frac{1}{2}\times\frac{1}{2}\left\{0.4805+1.9218+2\left(0.8396+1.2069+1.5694\right)\right\}$	M1	1.1b
	$=2.41$	A1	1.1b
		(3)	
(b)	$\int(\ln x)^2\ \mathrm{d}x = x(\ln x)^2-\int x\times\frac{2\ln x}{x}\ \mathrm{d}x$	M1 A1	3.1a 1.1b
	$=x(\ln x)^2-2\int\ln x\ \mathrm{d}x = x(\ln x)^2-2\left(x\ln x-\int\mathrm{d}x\right)$ $=x(\ln x)^2-2\int\ln x\ \mathrm{d}x = x(\ln x)^2-2x\ln x+2x$	dM1	2.1
	$\int_2^4(\ln x)^2\ \mathrm{d}x=\left[x(\ln x)^2-2x\ln x+2x\right]_2^4$ $=4(\ln 4)^2-2\times4\ln4+2\times4-\left(2(\ln2)^2-2\times2\ln2+2\times2\right)$ $=4(2\ln2)^2-16\ln2+8-2(\ln2)^2+4\ln2-4$	ddM1	2.1
	$=14(\ln 2)^2-12\ln2+4$	A1	1.1b
		(5)	
		(8 marks)	

Notes

(a)

B1: Correct strip width. May be implied by $\frac{1}{2}\times\frac{1}{2}\{....\}$ or $\frac{1}{4}\times\{....\}$

M1: Correct application of the trapezium rule.

Look for $\frac{1}{2}\times\text{"}h\text{"}\left\{0.4805+1.9218+2\left(0.8396+1.2069+1.5694\right)\right\}$ condoning slips in the digits.

The bracketing must be correct but it is implied by awrt 2.41

A1: 2.41 only. This is not awrt

(b)

M1: Attempts parts the correct way round to achieve $\alpha x(\ln x)^2-\beta\int\ln x\ \mathrm{d}x$ o.e.

May be unsimplified (see scheme). Watch for candidates who know or learn $\int\ln x\ \mathrm{d}x = x\ln x - x$

who may write $\int(\ln x)^2\ \mathrm{d}x=\int(\ln x)(\ln x)\ \mathrm{d}x=\ln x(x\ln x-x)-\int\frac{x\ln x-x}{x}\ \mathrm{d}x$

A1: Correct expression which may be unsimplified

dM1: Attempts parts again to (only condone coefficient errors) to achieve $\alpha x(\ln x)^2-\beta x\ln x\pm\gamma x$ o.e.

ddM1: Applies the limits 4 and 2 to an expression of the form $\pm\alpha x(\ln x)^2\pm\beta x\ln x\pm\gamma x$, subtracts and applies $\ln 4 = 2\ln 2$ at least once. Both M's must have been awarded

A1: Correct answer

....

It is possible to do $\int(\ln x)^2\ \mathrm{d}x$ via a substitution $u=\ln x$ but it is very similar.

M1 A1, dM1: $\int u^2\mathrm{e}^u\ \mathrm{d}u = u^2\mathrm{e}^u-\int 2u\mathrm{e}^u\ \mathrm{d}u, = u^2\mathrm{e}^u-2u\mathrm{e}^u\pm2\mathrm{e}^u$

ddM1: Applies appropriate limits and uses $\ln 4 = 2\ln 2$ at least once to an expression of the form $u^2\mathrm{e}^u-\beta u\mathrm{e}^u\pm\gamma\mathrm{e}^u$ Both M's must have been awarded

....

12. (a) Use the substitution $u = 1 + \sqrt{x}$ to show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\, \mathrm{d}x = \int_p^q \frac{2(u-1)^3}{u}\, \mathrm{d}u$$

where p and q are constants to be found.

(3)

(b) Hence show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\, \mathrm{d}x = A - B\ln 5$$

where A and B are constants to be found.

(4)

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Question	Scheme	Marks	AOs
12(a)	$u=1+\sqrt{x}\Rightarrow x=(u-1)^2\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}u}=2(u-1)$ or $u=1+\sqrt{x}\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}x^{-\frac{1}{2}}$	B1	1.1b
	$\int\frac{x}{1+\sqrt{x}}\mathrm{d}x=\int\frac{(u-1)^2}{u}2(u-1)\mathrm{d}u$ or $\int\frac{x}{1+\sqrt{x}}\mathrm{d}x=\int\frac{x}{u}\times2x^{\frac{1}{2}}\mathrm{d}u=\int\frac{2x^{\frac{3}{2}}}{u}\mathrm{d}u=\int\frac{2(u-1)^3}{u}\mathrm{d}u$	M1	2.1
	$\int_0^{16}\frac{x}{1+\sqrt{x}}\mathrm{d}x=\int_1^5\frac{2(u-1)^3}{u}\mathrm{d}u$	A1	1.1b
		(3)	
(b)	$2\int\frac{u^3-3u^2+3u-1}{u}\mathrm{d}u=2\int\left(u^2-3u+3-\frac{1}{u}\right)\mathrm{d}u=...$	M1	3.1a
	$=(2)\left[\frac{u^3}{3}-\frac{3u^2}{2}+3u-\ln u\right]$	A1	1.1b
	$=2\left[\frac{5^3}{3}-\frac{3(5)^2}{2}+3(5)-\ln5-\left(\frac{1}{3}-\frac{3}{2}+3-\ln1\right)\right]$	dM1	2.1
	$=\frac{104}{3}-2\ln5$	A1	1.1b
		(4)	
		(7 marks)	

Notes

(a)

B1: Correct expression for $\frac{\mathrm{d}x}{\mathrm{d}u}$ or $\frac{\mathrm{d}u}{\mathrm{d}x}$ (or u') or dx in terms of du or du in terms of dx

M1: Complete method using the given substitution.

This needs to be a correct method for their $\frac{\mathrm{d}x}{\mathrm{d}u}$ or $\frac{\mathrm{d}u}{\mathrm{d}x}$ leading to an integral in terms of u only (ignore any limits if present) so for each case you need to see:

$\frac{\mathrm{d}x}{\mathrm{d}u}=\mathrm{f}(u)\to\int\frac{x}{1+\sqrt{x}}\mathrm{d}x=\int\frac{(u-1)^2}{u}\mathrm{f}(u)\mathrm{d}u$

$\frac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{g}(x)\to\int\frac{x}{1+\sqrt{x}}\mathrm{d}x=\int\frac{x}{u}\times\frac{\mathrm{d}u}{\mathrm{g}(x)}=\int\mathrm{h}(u)\mathrm{d}u$. In this case you can condone slips with coefficients e.g. allow $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}x^{-\frac{1}{2}}\to\int\frac{x}{1+\sqrt{x}}\mathrm{d}x=\int\frac{x}{u}\times\frac{x^{\frac{1}{2}}}{2}\mathrm{d}u=\int\mathrm{h}(u)\mathrm{d}u$

but not $\frac{du}{dx}=\frac{1}{2}x^{-\frac{1}{2}} \to \int \frac{x}{1+\sqrt{x}}dx = \int \frac{x}{u}\times\frac{x^{-\frac{1}{2}}}{2}du = \int h(u)du$

A1: All correct with correct limits and no errors. The "du" must be present but may have been omitted along the way but it must have been seen at least once before the final answer. The limits can be seen as part of the integral or stated separately.

(b)

M1: Realises the requirement to cube the bracket and divide through by u and makes progress with the integration to obtain at least 3 terms of the required form e.g. 3 from ku^3, ku^2, ku, $k\ln u$

A1: Correct integration. This mark can be scored with the "2" still outside the integral or even if it has been omitted. But if the "2" has been combined with the integrand, the integration must be correct.

dM1: Completes the process by applying their "changed" limits and subtracts the right way round **Depends on the first method mark.**

A1: Cao (Allow equivalents for $\frac{104}{3}$ e.g. $\frac{208}{6}$)

14.

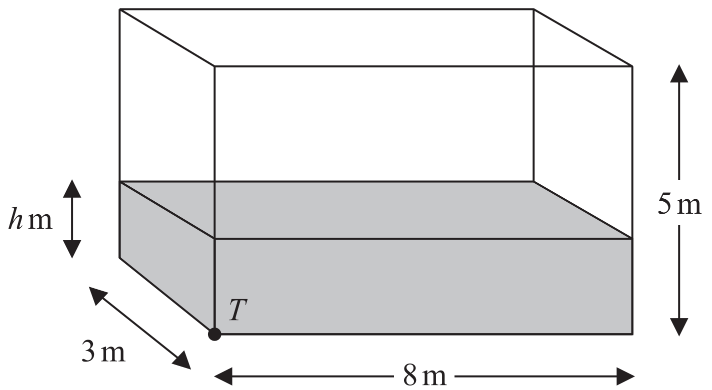

Figure 5

Water flows at a constant rate into a large tank.

The tank is a cuboid, with all sides of negligible thickness.

The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.

There is a tap at a point T at the bottom of the tank, as shown in Figure 5.

At time t minutes after the tap has been opened

- the depth of water in the tank is h metres
- water is flowing into the tank at a constant rate of $0.48\,\text{m}^3$ per minute
- water is modelled as leaving the tank through the tap at a rate of $0.1h\,\text{m}^3$ per minute

(a) Show that, according to the model,

$$1200\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h$$

(4)

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model,

$$h = A + B\mathrm{e}^{-kt}$$

where A, B and k are constants to be found.

(6)

Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

(2)

Question	Scheme	Marks	AOs
14(a)	$\frac{\mathrm{d}V}{\mathrm{d}t} = 0.48 - 0.1h$	B1	3.1b
	$V = 24h \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}h} = 24$ or $\frac{\mathrm{d}h}{\mathrm{d}V} = \frac{1}{24}$	B1	3.1b
	$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \times \frac{\mathrm{d}h}{\mathrm{d}V} = \frac{0.48 - 0.1h}{24}$ or e.g. $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}h}\frac{\mathrm{d}h}{\mathrm{d}t} \Rightarrow 0.48 - 0.1h = 24\frac{\mathrm{d}h}{\mathrm{d}t}$	M1	2.1
	$1200\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h$ *	A1*	1.1b
		(4)	
(b)	$1200\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h \Rightarrow \int \frac{1200}{24-5h}\mathrm{d}h = \int \mathrm{d}t$ $\Rightarrow$ *e.g.* $\alpha \ln(24-5h) = t(+c)$ oe or $1200\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}h} = \frac{1200}{24-5h}$ $\Rightarrow$ *e.g.* $t(+c) = \alpha \ln(24-5h)$ oe	M1	3.1a
	$t = -240\ln(24-5h)(+c)$ oe	A1	1.1b
	$t = 0, h = 2 \Rightarrow 0 = -240\ln(24-10) + c \Rightarrow c = ...(240\ln 14)$	M1	3.4
	$t = 240\ln(14) - 240\ln(24-5h)$	A1	1.1b
	$t = 240\ln\frac{14}{24-5h} \Rightarrow \frac{t}{240} = \ln\frac{14}{24-5h} \Rightarrow e^{\frac{t}{240}} = \frac{14}{24-5h}$ $\Rightarrow 14e^{-\frac{t}{240}} = 24 - 5h \Rightarrow h = ...$	ddM1	2.1
	$h = 4.8 - 2.8e^{-\frac{t}{240}}$ oe e.g. $h = \frac{24}{5} - \frac{14}{5}e^{-\frac{t}{240}}$	A1	3.3
		(6)	
(c)	**Examples:** • As $t \to \infty, e^{-\frac{t}{240}} \to 0$ • When $h > 4.8$, $\frac{\mathrm{d}V}{\mathrm{d}t} < 0$ • Flow in = flow out at max h so $0.1h = 4.8 \rightarrow h = 4.8$ • As $e^{-\frac{t}{240}} > 0$, $h < 4.8$ • $h = 5 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = -0.02$ or $\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{1}{1200}$ • $\frac{\mathrm{d}h}{\mathrm{d}t} = 0 \Rightarrow h = 4.8$ • $h = 5 \Rightarrow 4.8 - 2.8e^{-\frac{t}{240}} = 5 \Rightarrow e^{-\frac{t}{240}} < 0$	M1	3.1b
	• The limit for h (according to the model) is 4.8m and the tank is 5m high so the tank will never become full • If $h = 5$ the tank would be emptying so can never be full • The equation can't be solved when $h = 5$	A1	3.2a

		(2)	
			(12 marks)

Notes

(a)

B1: Identifies the correct expression for $\frac{\mathrm{d}V}{\mathrm{d}t}$ according to the model

B1: Identifies the correct expression for $\frac{\mathrm{d}V}{\mathrm{d}h}$ according to the model

M1: Applies $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \times \frac{\mathrm{d}h}{\mathrm{d}V}$ or equivalent correct formula with their $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $\frac{\mathrm{d}V}{\mathrm{d}h}$ which may be implied by their working

A1*: Correct equation obtained with no errors

Note that: $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.48 - 0.1h \Rightarrow \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{0.48 - 0.1h}{24} \Rightarrow 1200\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h$ * scores B1B0M0A0. There must be clear evidence where the "24" comes from and evidence of the correct chain rule being applied.

(b)

M1: Adopts a correct strategy by separating the variables correctly or rearranges to obtain $\frac{\mathrm{d}t}{\mathrm{d}h}$ correctly in terms of *h* **and** integrates to obtain $t = \alpha \ln(24 - 5h)(+c)$ or equivalent (condone missing brackets around the "24 – 5*h*") and + *c* not required for this mark.

A1: Correct equation in any form and + *c* not required. Do not condone missing brackets unless they are implied by subsequent work.

M1: Substitutes *t* = 0 and *h* = 2 to find their constant of integration (there must have been some attempt to integrate)

A1: Correct equation in any form

ddM1: Uses fully correct log work to obtain *h* in terms of *t*.

This depends on <u>both</u> previous method marks.

A1: Correct equation

Note that the marks may be earned in a different order e.g.:

$$t + c = -240\ln(24 - 5h) \Rightarrow -\frac{t}{240} + d = \ln(24 - 5h) \Rightarrow Ae^{-\frac{t}{240}} = 24 - 5h$$

$$t = 0, h = 2 \Rightarrow A = 14 \Rightarrow 14e^{-\frac{t}{240}} = 24 - 5h \Rightarrow h = 4.8 - 2.8e^{-\frac{t}{240}}$$

Score as M1 A1 as in main scheme then

M1: Correct work leading to $Ae^{\alpha t} = 24 - 5h$ (must have a constant "A")

A1: $Ae^{-\frac{t}{240}} = 24 - 5h$

ddM1: Uses *t* = 0, *h* = 2 in an expression of the form above to find *A*

A1: $h = 4.8 - 2.8e^{-\frac{t}{240}}$

(c)

M1: See scheme for some examples

A1: Makes a correct interpretation for their method.

There must be no incorrect working or contradictory statements.

This is not a follow through mark and if their equation in (b) is used it must be correct.

4. (a) Express $\lim_{\delta x \to 0} \sum_{x=2.1}^{6.3} \frac{2}{x}\delta x$ as an integral.

(1)

(b) Hence show that

$$\lim_{\delta x \to 0} \sum_{x=2.1}^{6.3} \frac{2}{x}\delta x = \ln k$$

where k is a constant to be found.

(2)

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Question	Scheme	Marks	AOs
4 (a)	$\lim_{\delta x \to 0} \sum_{x=2.1}^{6.3} \frac{2}{x} \delta x = \int_{2.1}^{6.3} \frac{2}{x} \, dx$	B1	1.2
		(1)	
(b)	$= \left[2\ln x\right]_{2.1}^{6.3} = 2\ln 6.3 - 2\ln 2.1$	M1	1.1b
	$= \ln 9$ CSO	A1	1.1b
		(2)	
			(3 marks)
Notes:			

Mark (a) and (b) as one

(a)

B1: States that $\int_{2.1}^{6.3} \frac{2}{x} \, dx$ or equivalent such as $2\int_{2.1}^{6.3} x^{-1} \, dx$ but must include the limits and the dx.

Condone $dx \leftrightarrow \delta x$ as it is very difficult to tell one from another sometimes

(b)

M1: Know that $\int \frac{1}{x} dx = \ln x$ and attempts to apply the limits (either way around)

Condone $\int \frac{2}{x} dx = p \ln x$ (including $p = 1$) or $\int \frac{2}{x} dx = p \ln qx$ as long as the limits are applied.

Also be aware that $\int \frac{2}{x} dx = \ln x^2$, $\int \frac{2}{x} dx = 2\ln|x| + c$ and $\int \frac{2}{x} dx = 2\ln cx$ o.e. are also correct

$\left[p \ln x\right]_{2.1}^{6.3} = p\ln 6.3 - p\ln 2.1$ is sufficient evidence to award this mark

A1: CSO ln 9 . Also answer = $\ln 3^2$ so $k = 9$ is fine. Condone $\ln|9|$

The method mark must have been awarded. Do not accept answers such as $\ln\frac{39.69}{4.41}$

Note that solutions appearing from "rounded" decimal work when taking lns should not score the final mark. It is a "show that" question

E.g. $\left[2\ln x\right]_{2.1}^{6.3} = 2\ln 6.3 - 2\ln 2.1 = 2.197 = \ln k \Rightarrow k = e^{2.197} = 8.998 = 9$

12. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Show that

$$\int_1^{e^2} x^3 \ln x \, dx = ae^8 + b$$

where a and b are rational constants to be found.

(5)

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Question	Scheme	Marks	AOs
12	$\int x^3 \ln x \, dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4} \times \frac{1}{x} \, dx$	M1	1.1b
	$= \frac{x^4}{4}\ln x - \frac{x^4}{16}(+c)$	M1 A1	1.1b 1.1b
	$\int_1^{e^2} x^3 \ln x \, dx = \left[\frac{x^4}{4}\ln x - \frac{x^4}{16}\right]_1^{e^2} = \left(\frac{e^8}{4}\ln e^2 - \frac{e^8}{16}\right) - \left(-\frac{1^4}{16}\right)$	M1	2.1
	$= \frac{7}{16}e^8 + \frac{1}{16}$	A1	1.1b
		(5)	
			(5 marks)
Notes:			

M1: Integrates by parts the right way round.

Look for $kx^4 \ln x - \int kx^4 \times \frac{1}{x} \, dx$ o.e. with $k > 0$. Condone a missing dx

M1: Uses a correct method to integrate an expression of the form $\int kx^4 \times \frac{1}{x} \, dx \to c\,x^4$

A1: $\int x^3 \ln x \, dx = \frac{x^4}{4}\ln x - \frac{x^4}{16}$ $(+\,c)$ which may be left unsimplified

M1: Attempts to substitute 1 and e^2 into an expression of the form $\pm px^4 \ln x \pm qx^4$, subtracts and uses $\ln e^2 = 2$ (which may be implied).

A1: $\frac{7}{16}e^8 + \frac{1}{16}$ o.e. Allow $0.4375e^8 + 0.0625$ or uncancelled fractions. NOT ISW: $7e^8 + 1$ is A0

....

You may see attempts where substitution has been attempted.

E.g. $u = \ln x \Rightarrow x = e^u$ and $\frac{dx}{du} = e^u$

M1: Attempts to integrate the correct way around condoning slips on the coefficients

$$\int x^3 \ln x \, dx = \int e^{4u} u \, du = \frac{e^{4u}}{4}u - \int \frac{e^{4u}}{4} \, du$$

M1 A1: $\int x^3 \ln x \, dx = \frac{e^{4u}}{4}u - \frac{e^{4u}}{16}(+c)$

M1 A1: Substitutes 0 and 2 into an expression of the form $\pm pue^{4u} \pm qe^{4u}$ and subtracts

....

It is possible to use integration by parts "the other way around"

To do this, candidates need to know or use $\int \ln x \, dx = x\ln x - x$

FYI $I = \int x^3 \ln x \, dx = x^3(x\ln x - x) - \int (x\ln x - x) \times 3x^2 \, dx = x^3(x\ln x - x) - 3I + \frac{3}{4}x^4$

Hence $4I = x^4 \ln x - \frac{1}{4}x^4 \Rightarrow I = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4$

Score M1 for a full attempt at line 1 (condoning bracketing and coefficient slips) followed by M 1 for line 2 where terms in I o.e. to form the answer.

16.

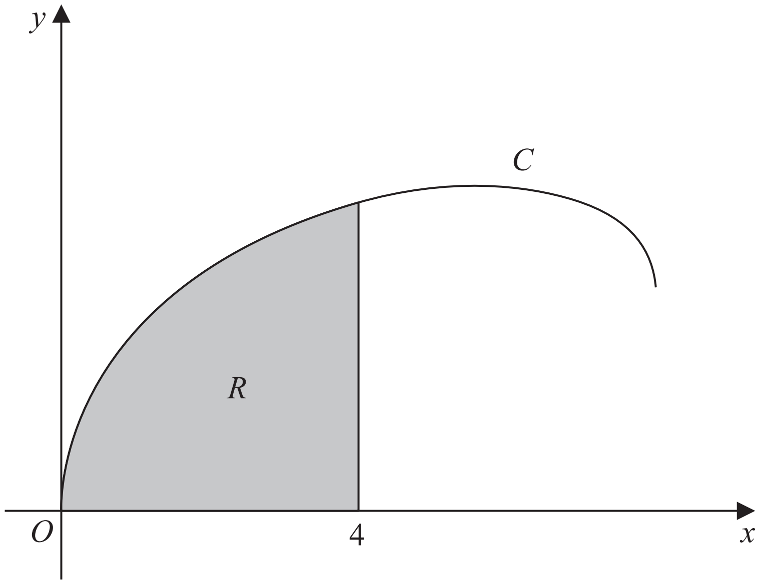

Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 8\sin^2 t \qquad y = 2\sin 2t + 3\sin t \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

The region R, shown shaded in Figure 6, is bounded by C, the x-axis and the line with equation $x = 4$

(a) Show that the area of R is given by

$$\int_0^a \left(8 - 8\cos 4t + 48\sin^2 t\cos t\right)\mathrm{d}t$$

where a is a constant to be found.

(5)

(b) Hence, using algebraic integration, find the exact area of R.

(4)

Question	Scheme	Marks	AOs
16 (a)	Attempts $y.\frac{\mathrm{d}x}{\mathrm{d}t}=(2\sin 2t+3\sin t)\times 16\sin t\cos t$ and uses $\sin 2t=2\sin t\cos t$	M1	2.1
	Correct expanded integrand. **Usually** for one of $(R)=\int \underline{\underline{48\sin^2 t\cos t+16\sin^2 2t}}\,\mathrm{d}t$ $(R)=\int \underline{\underline{48\sin^2 t\cos t+64\sin^2 t\cos^2 t}}\,\mathrm{d}t$ $(R)=\int \underline{\underline{24\sin 2t\sin t+16\sin^2 2t}}\,\mathrm{d}t$	A1	1.1b
	Attempts to use $\cos 4t=1-2\sin^2 2t=\left(1-8\sin^2 t\cos^2 t\right)$	M1	1.1b
	$R=\int_0^a 8-8\cos 4t+48\sin^2 t\cos t\,\mathrm{d}t$ *	A1*	2.1
	Deduces $a=\frac{\pi}{4}$	B1	2.2a
		(5)	
(b)	$\int 8-8\cos 4t+48\sin^2 t\cos t\,\mathrm{d}t=8t-2\sin 4t+16\sin^3 t$	M1 A1	2.1 1.1b
	$\left[8t-2\sin 4t+16\sin^3 t\right]_0^{\frac{\pi}{4}}=2\pi+4\sqrt{2}$	M1 A1	2.1 1.1b
		(4)	
		(9 marks)	
Notes:			

(a) Condone work in another variable, say $\theta \leftrightarrow t$ if used consistently for the first 3 marks

M1: For the key step in attempting $y.\frac{\mathrm{d}x}{\mathrm{d}t}=(2\sin 2t+3\sin t)\times 16\sin t\cos t$ with an attempt to use $\sin 2t=2\sin t\cos t$ Condone slips in finding $\frac{\mathrm{d}x}{\mathrm{d}t}$ but it must be of the form $k\sin t\cos t$

E.g. I $\quad y.\frac{\mathrm{d}x}{\mathrm{d}t}=(2\sin 2t+3\sin t)\times k\sin t\cos t=(4\sin t\cos t+3\sin t)\times k\sin t\cos t$

E.g. II $\quad y.\frac{\mathrm{d}x}{\mathrm{d}t}=(2\sin 2t+3\sin t)\times k\sin t\cos t=(2\sin 2t+3\sin t)\times\frac{k}{2}\sin 2t$

A1: A correct (expanded) integrand in t. Don't be concerned by the absence of $\int$ or dt or limits

$(R)=\int \underline{\underline{48\sin^2 t\cos t+16\sin^2 2t}}\,\mathrm{d}t$ or $(R)=\int \underline{\underline{48\sin^2 t\cos t+64\sin^2 t\cos^2 t}}\,\mathrm{d}t$

but watch for other correct versions such as $(R)=\int \underline{\underline{24\sin 2t\sin t+16\sin^2 2t}}\,\mathrm{d}t$

M1: Attempts to use $\cos 4t = \pm 1 \pm 2\sin^2 2t$ to get the integrand in the correct form.

If they have the form $P\sin^2 2t$ it is acceptable to write $P\sin^2 2t = \frac{P}{2}(\pm 1 \pm \cos 4t)$

If they have the form $Q\sin^2 t\cos^2 t$ sight and use of $\sin 2t$ and/or $\cos 2t$ will usually be seen first. There are many ways to do this, below is such an example

$$Q\sin^2 t\cos^2 t = Q\left(\frac{1-\cos 2t}{2}\right)\left(\frac{1+\cos 2t}{2}\right) = Q\left(\frac{1-\cos^2 2t}{4}\right) = Q\left(\frac{1}{4}-\frac{\cos^2 2t}{4}\right) = Q\left(\frac{1}{4}-\frac{1+\cos 4t}{8}\right)$$

Allow candidates to start with the given answer and work backwards using the same rules. So expect to see $\cos 4t = \pm 1 \pm 2\times\sin^2 2t$ or $\cos 4t = \pm 2\times\cos^2 2t \pm 1$ before double angle identities for $\sin 2t$ or $\cos 2t$ are used.

A1*: Proceeds to the given answer with correct working. The order of the terms is not important. Ignore limits for this mark. The integration sign and the dt must be seen on their final answer. If they have worked backwards there must be a concluding statement to the effect that they know that they have shown it. The integration sign and the dt must also be seen

E.g. Reaches $\int 48\sin^2 t\cos t + 64\sin^2 t\cos^2 t \, dt$

Answer is $\int 8 - 8\cos 4t + 48\sin^2 t\cos t \, dt$

$$= \int 8 - 8(1 - 2\sin^2 2t) + 48\sin^2 t\cos t \, dt$$

$$= \int 16\sin^2 2t + 48\sin^2 t\cos t \, dt$$

$$= \int 64\sin^2 t\cos^2 t + 48\sin^2 t\cos t \, dt$$

which is the same, ✓

B1: Deduces $a = \frac{\pi}{4}$. It may be awarded from the upper limit and can be awarded from (b)

(b)

M1: For the key process in using a correct approach to integrating the trigonometric terms. May be done separately.
There may be lots of intermediate steps (e.g. let $u = \sin t$).
There are other more complicated methods so look carefully at what they are doing.

$\int 8 - 8\cos 4t + 48\sin^2 t\cos t \, dt = ... \pm P\sin 4t \pm Q\sin^3 t$ where P and Q are constants

A1: $\int 8 - 8\cos 4t + 48\sin^2 t\cos t \, dt = 8t - 2\sin 4t + 16\sin^3 t (+c)$

If they have written $16\sin^3 t$ as $16\sin t^3$ only award if further work implies a correct answer. Similarly, 8t may be written as 8x. Award if further work implies 8t, e.g. substituting in their limits. Do not penalise this sort of slip at all, these are intermediate answers.

M1: Uses the limits their a and 0 where $a = \frac{\pi}{6}, \frac{\pi}{4}$ or $\frac{\pi}{3}$ in an expression of the form $kt \pm P\sin 4t \pm Q\sin^3 t$ leading to an exact answer. Ignore evidence at lower limit as terms are 0

A1: CSO $2\pi + 4\sqrt{2}$ or exact **simplified** equivalent such as $2\pi + \frac{8}{\sqrt{2}}$ or $2\pi + \sqrt{32}$.

Be aware that $\int_0^{\frac{\pi}{4}} 8 - 8\cos 4t + 48\sin^2 t\cos t \, dt = 8t + \lambda\sin 4t + 16\sin^3 t (+c)$ would lead to the correct answer but would only score M1 A0 M1 A0

5. The table below shows corresponding values of x and y for $y = \log_3 2x$

The values of y are given to 2 decimal places as appropriate.

x	3	4.5	6	7.5	9
y	1.63	2	2.26	2.46	2.63

(a) Using the trapezium rule with all the values of y in the table, find an estimate for

$$\int_3^9 \log_3 2x \, dx$$

(3)

Using your answer to part (a) and making your method clear, estimate

(b) (i) $\int_3^9 \log_3 (2x)^{10} \, dx$

(ii) $\int_3^9 \log_3 18x \, dx$

(3)

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question	Scheme	Marks	AOs
5(a)	States or uses $h=1.5$	B1	1.1a
	Full attempt at the trapezium rule $=\frac{...}{2}\{1.63+2.63+2\times(2+2.26+2.46)\}$	M1	1.1b
	$=$ awrt 13.3 or $\frac{531}{40}$	A1	1.1b
		(3)	
(b)(i)	$\int_3^9 \log_3(2x)^{10}\,dx = 10\times"13.3" = \text{awrt}133$ or e.g. $\frac{531}{4}$	B1ft	2.2a
(ii)	$\int_3^9 \log_3 18x\,dx = \int_3^9 \log_3(9\times 2x)\,dx = \int_3^9 2+\log_3 2x\,dx$ $=[2x]_3^9 + \int_3^9 \log_3 2x\,dx = 18-6+\int_3^9 \log_3 2x\,dx = ...$	M1	3.1a
	Awrt 25.3 or $\frac{1011}{40}$	A1ft	1.1b
		(3)	
			(6 marks)

Notes:

(a)

B1: States or uses $h=1.5$

M1: A full attempt at the trapezium rule.

Look for $\frac{\text{their } h}{2}\{1.63+2.63+2\times(2+2.26+2.46)\}$ but condone copying slips

Note that $\frac{\text{their } h}{2}1.63+2.63+2\times(2+2.26+2.46)$ scores M0 unless the missing brackets are recovered or implied by their answer. You may need to check.

Allow this mark if they add the areas of individual trapezia e.g.

$\frac{\text{their } h}{2}\{1.63+2\}+\frac{\text{their } h}{2}\{2+2.26\}+\frac{\text{their } h}{2}\{2.26+2.46\}+\frac{\text{their } h}{2}\{2.46+2.63\}$

Condone copying slips but must be a complete method using all the trapezia.

A1: awrt 13.3 (Note full accuracy is 13.275) or exact equivalent.

Note that the calculator answer is 13.324 so you must see correct working to award awrt 13.3

Use of $h=-1.5$ leading to a negative area can score B1M1A0 but allow full marks if then stated as positive.

(b)(i)

B1ft: Deduces that $\int_3^9 \log_3(2x)^{10}\,dx = 10\times"13.3" = \text{awrt } 133$

FT on their 13.3 look for 3sf accuracy but follow through on e.g. their rounded answer to part (a) so if 13 was their answer to part (a) then allow 130 here **following a correct method**.

A correct method must be seen here but a minimum is e.g. 10 × "13.3" = "133"

Note that $\int_3^9 \log_3(2x)^{10}\,dx = 133.2414316...$ so a correct method must be seen to award marks.

Attempts to apply the trapezium rule again in any way score M0 as the instruction in the question was to use the answer to part (a).

(b)(ii)

M1: Shows correct log work to relate the given question to part (a)

Must reach as far as e.g. $[2x]_3^9 + \int_3^9 \log_3 2x \, dx = ...$ with correct use of limits on $[2x]_3^9$ which may be implied or equivalent work e.g. finds the area of the rectangle as 2×6

A1ft: **Correct working** followed by awrt 25.3 but ft on their 13.3 so allow for 12 + their answer to part (a) **following correct work** as shown.

Note that $\int_3^9 \log_3 18x \, dx = 25.32414...$ **so a correct method must be seen to award marks**.

Some examples of an acceptable method are:

$$\int_3^9 \log_3 18x \, dx = \int_3^9 \log_3 (9 \times 2x) \, dx = \int_3^9 2 + \log_3 2x \, dx = 6 \times 2 + "13.3" = 25.3$$

$$\int_3^9 \log_3 18x \, dx = \int_3^9 \log_3 (9 \times 2x) \, dx = \int_3^9 2 + \log_3 2x \, dx = 12 + "13.3" = 25.3$$

$$\int_3^9 \log_3 18x \, dx = \int_3^9 \log_3 (9 \times 2x) \, dx = \int_3^9 2 + \log_3 2x \, dx = [2x]_3^9 + \int_3^9 \log_3 2x \, dx = 25.3$$

BUT just $12 + "13.3" = 25.3$ **scores M0**

Attempts to apply the trapezium rule again in any way score M0 as the instruction in the question was to use the answer to part (a).

14. (a) Express $\dfrac{3}{(2x-1)(x+1)}$ in partial fractions.

(3)

When chemical A and chemical B are mixed, oxygen is produced.

A scientist mixed these two chemicals and measured the total volume of oxygen produced over a period of time.

The total volume of oxygen produced, $V\,\text{m}^3$, t hours after the chemicals were mixed, is modelled by the differential equation

$$\frac{dV}{dt} = \frac{3V}{(2t-1)(t+1)} \qquad V \geqslant 0 \qquad t \geqslant k$$

where k is a constant.

Given that exactly 2 hours after the chemicals were mixed, a total volume of $3\,\text{m}^3$ of oxygen had been produced,

(b) solve the differential equation to show that

$$V = \frac{3(2t-1)}{(t+1)}$$

(5)

The scientist noticed that

- there was a **time delay** between the chemicals being mixed and oxygen being produced
- there was a **limit** to the total volume of oxygen produced

Deduce from the model

(c) (i) the **time delay** giving your answer in minutes,

(ii) the **limit** giving your answer in m^3

(2)

Question	Scheme	Marks	AOs
14(a)	$\frac{3}{(2x-1)(x+1)}=\frac{A}{2x-1}+\frac{B}{x+1}\Rightarrow A=...,\ B=...$	M1	1.1b
	Either $A=2$ or $B=-1$	A1	1.1b
	$\frac{3}{(2x-1)(x+1)}=\frac{2}{2x-1}-\frac{1}{x+1}$	A1	1.1b
		(3)	
(b)	$\int \frac{1}{V}\mathrm{d}V=\int \frac{3}{(2t-1)(t+1)}\ \mathrm{d}t$	B1	1.1a
	$\int \frac{2}{2t-1}-\frac{1}{t+1}\mathrm{d}t=...\ln(2t-1)-...\ln(t+1)(+c)$	M1	3.1a
	$\ln V=\ln(2t-1)-\ln(t+1)(+c)$	A1ft	1.1b
	Substitutes $t=2, V=3\Rightarrow c=(\ln 3)$	M1	3.4
	$\ln V=\ln(2t-1)-\ln(t+1)+\ln 3$ $V=\frac{3(2t-1)}{(t+1)}$ *	A1*	2.1
		(5)	
	(b) Alternative separation of variables:		
	$\int \frac{1}{3V}\mathrm{d}V=\int \frac{1}{(2t-1)(t+1)}\ \mathrm{d}t$	B1	1.1a
	$\frac{1}{3}\int \frac{2}{2t-1}-\frac{1}{t+1}\mathrm{d}t=...\ln(2t-1)-...\ln(t+1)(+c)$	M1	3.1a
	$\frac{1}{3}\ln 3V=\frac{1}{3}\ln(2t-1)-\frac{1}{3}\ln(t+1)(+c)$	A1ft	1.1b
	Substitutes $t=2, V=3\Rightarrow c=\left(\frac{1}{3}\ln 3\right)$	M1	3.4
	$\frac{1}{3}\ln V=\frac{1}{3}\ln(2t-1)-\frac{1}{3}\ln(t+1)+\frac{1}{3}\ln 3$ $V=\frac{3(2t-1)}{(t+1)}$ *	A1*	2.1
		(5)	
(c)	(i) 30 (minutes)	B1	3.2a
	(ii) 6 (m^3)	B1	3.4
		(2)	
			(10 marks)
Notes:			

(a)

M1: Correct method of partial fractions leading to values for their A and B

E.g. substitution: $\frac{3}{(2x-1)(x+1)}=\frac{A}{2x-1}+\frac{B}{x+1}\Rightarrow 3=A(x+1)+B(2x-1)\Rightarrow A=...,B=...$

Or compare coefficients $\frac{3}{(2x-1)(x+1)}=\frac{A}{2x-1}+\frac{B}{x+1}\Rightarrow 3=x(A+2B)+A-B\Rightarrow A=...,B=...$

Note that $\frac{3}{(2x-1)(x+1)}=\frac{A}{2x-1}+\frac{B}{x+1}\Rightarrow 3=A(2x-1)+B(x+1)\Rightarrow A=...,B=...$ scores M0

A1: Correct value for "A" or "B"

A1: Correct partial fractions not just values for "A" and "B". $\dfrac{2}{2x-1}-\dfrac{1}{x+1}$ or e.g. $\dfrac{2}{2x-1}+\dfrac{-1}{x+1}$

Must be seen as **fractions** but if not stated here, allow if the correct fractions appear later.

(b)

B1: Separates variables $\displaystyle\int \frac{1}{V}\,\mathrm{d}V = \int \frac{3}{(2t-1)(t+1)}\,\mathrm{d}t$. May be implied by later work.

Condone omission of the integral signs but the dV and dt must be in the correct positions if awarding this mark in isolation but they may be implied by subsequent work.

M1: Correct attempt at integration of the partial fractions.

Look for $...\ln(2t-1)+...\ln(t+1)$ where … are constants.

Condone missing brackets around the ($2t$ – 1) and/or the (t + 1) for this mark

A1ft: Fully correct equation following through their A and B **only**.

No requirement for $+c$ here.

The brackets around the ($2t$ – 1) and/or the (t + 1) must be seen or implied for this mark

M1: Attempts to find "c" or e.g. "ln k" using $t=2,\ V=3$ following an attempt at integration.

Condone poor algebra as long as $t=2,\ V=3$ is used to find a value of their constant.

Note that the constant may be found immediately after integrating or e.g. after the ln's have been combined.

A1*: Correct processing leading to the given answer $V=\dfrac{3(2t-1)}{(t+1)}$

Alternative:

B1: Separates variables $\displaystyle\int \frac{1}{3V}\,\mathrm{d}V = \int \frac{1}{(2t-1)(t+1)}\,\mathrm{d}t$. May be implied by later work.

Condone omission of the integral signs but the dV and dt must be in the correct positions if awarding this mark in isolation but they may be implied by subsequent work.

M1: Correct attempt at integration of the partial fractions.

Look for $...\ln(2t-1)+...\ln(t+1)$ where … are constants.

Condone missing brackets around the ($2t$ – 1) and/or the (t + 1) for this mark

A1ft: Fully correct equation following through their A and B **only**.

No requirement for $+c$ here.

The brackets around the ($2t$ – 1) and/or the (t + 1) must be seen or implied for this mark

M1: Attempts to find "c" or e.g. "ln k" using $t=2,\ V=3$ following an attempt at integration.

Condone poor algebra as long as $t=2,\ V=3$ is used to find a value of their constant.

Note that the constant may be found immediately after integrating or e.g. after the ln's have been combined.

A1*: Correct processing leading to the given answer $V=\dfrac{3(2t-1)}{(t+1)}$

(Note the working may look like this:

$$\frac{1}{3}\ln 3V = \frac{1}{3}\ln(2t-1)-\frac{1}{3}\ln(t+1)+c,\ \frac{1}{3}\ln 9=\frac{1}{3}\ln(3)-\frac{1}{3}\ln 3+c,\ c=\frac{1}{3}\ln 9$$

$$\ln 3V = \ln\frac{9(2t-1)}{(t+1)} \Rightarrow 3V=\frac{9(2t-1)}{(t+1)} \Rightarrow V=\frac{3(2t-1)}{(t+1)}*)$$

Note that B0M1A1M1A1 is not possible in (b) as the B1 must be implied if all the other marks have been awarded.

Note also that some candidates may use different variables in (b) e.g.

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3y}{(2x-1)(x+1)} \Rightarrow \displaystyle\int \frac{1}{y}\,\mathrm{d}y = \int \frac{3}{(2x-1)(x+1)}\,\mathrm{d}x$ etc. In such cases you should award marks for equivalent work but they must revert to the given variables at the end to score the final mark.

Also if e.g. a "t" becomes an "x" within their working but is recovered allow full marks.

(c)

B1: Deduces 30 minutes. Units not required so just look for 30 but allow equivalents e.g. ½ an hour.
If units are given they must be correct so do not allow e.g. 30 hours.

B1: Deduces 6 m^3. Units not required so just look for 6. Condone $V < 6$ or $V \leq 6$
If units are given they must be correct so do not allow e.g. 6 m.