Fp1Ch9 XMQs and MS

(Total: 76 marks)

1. FP1_Sample Q3 . 14 marks - FP1ch9 Reducible differential equations
2. FP1_Specimen Q8 . 15 marks - FP1ch9 Reducible differential equations
3. FP1_2019 Q6 . 17 marks - FP1ch9 Reducible differential equations
4. FP1_2021 Q8 . 17 marks - FP1ch8 Numerical methods
5. FP1_2022 Q9 . 13 marks - FP1ch9 Reducible differential equations

3. A vibrating spring, fixed at one end, has an external force acting on it such that the centre of the spring moves in a straight line. At time t seconds, $t \geqslant 0$, the displacement of the centre C of the spring from a fixed point O is x micrometres.

The displacement of C from O is modelled by the differential equation

$$t^2 \frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + (2 + t^2)x = t^4 \qquad \text{(I)}$$

(a) Show that the transformation $x = tv$ transforms equation (I) into the equation

$$\frac{d^2v}{dt^2} + v = t \qquad \text{(II)}$$

(5)

(b) Hence find the general equation for the displacement of C from O at time t seconds.

(7)

(c) (i) State what happens to the displacement of C from O as t becomes large.

(ii) Comment on the model with reference to this long term behaviour.

(2)

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Question	Scheme	Marks	AOs
3(a)	Use of $x = tv$ to give $\frac{\mathrm{d}x}{\mathrm{d}t} = v + t\frac{\mathrm{d}v}{\mathrm{d}t}$	M1	1.1b
	Hence $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\mathrm{d}v}{\mathrm{d}t} + t\frac{\mathrm{d}^2v}{\mathrm{d}t^2}$	M1	2.1
		A1	1.1b
	Uses $t^2\left(\text{their 2}^{\text{nd}}\text{derivative}\right) - 2t\left(\text{their 1}^{\text{st}}\text{derivative}\right) + (2+t^2)x = t^4$ and simplifies LHS	M1	2.1
	$\left(t^3\frac{\mathrm{d}^2v}{\mathrm{d}t^2} + t^3v = t^4 \text{ leading to}\right)$ $\frac{\mathrm{d}^2v}{\mathrm{d}t^2} + v = t$ *	A1*	1.1b
		(5)	
(b)	Solve $\lambda^2 + 1 = 0$ to give $\lambda^2 = -1$	M1	1.1b
	$v = A\cos t + B\sin t$	A1ft	1.1b
	Particular Integral is $v = kt + l$	B1	2.2a
	$\frac{\mathrm{d}v}{\mathrm{d}t} = k$ and $\frac{\mathrm{d}^2v}{\mathrm{d}t^2} = 0$ and solve $0 + kt + l = t$ to give $k = 1,\ l = 0$	M1	1.1b
	Solution: $v = A\cos t + B\sin t + t$	A1	1.1b
	Displacement of C from O is given by $x = tv = \ldots$	M1	3.4
	$x = t\left(A\cos t + B\sin t + t\right)$	A1	2.2a
		(7)	
(c)(i)	For large t, the displacement gets very large (and positive)	B1	3.2a
(ii)	Model suggests midpoint of spring moving relative to fixed point has large displacement when t is large, which is unrealistic. The spring may reach elastic limit / will break	B1	3.5a
		(2)	
			(14 marks)

Question 3 notes:	
(a)	
M1:	Uses product rule to obtain first derivative
M1:	Continues to differentiate again, with product rule and chain rule as appropriate, in order to establish the second derivative
A1:	Correct second derivative. Accept equivalent expressions
M1:	Shows clearly the substitution into the given equation in order to form the new equation and gathers like terms
A1*:	Fully correct solution leading to the given answer
(b)	Accept variations on symbols for constants throughout
M1:	Form and solve a quadratic Auxiliary Equation
A1ft:	Correct form of the Complementary Function for their solutions to the AE
B1:	Deduces the correct form for the Particular Integral (note $v = mt^2 + kt + l$ is fine)
M1:	Differentiates their Particular Integral and substitutes their derivatives into the equation to find the constants (m = 0 if used)
A1:	Correct general solution for equation (II)
M1:	Links the solution to equation (II) to the solution of the model equation correctly to find the displacement equation
A1:	Deduces the correct general solution for the displacement
(c)(i)	
B1:	States that for large t the displacement is large o.e. Accept e.g. as $t \to \infty$, $x \to \infty$
(c)(ii)	
B1:	Reflect on the context of the original problem. Accept 'model unrealistic' / 'spring will break'

8. A particle is moving in a straight line. The displacement, x m, of the particle relative to a fixed origin O, after it has been moving for t seconds, is modelled by the differential equation

$$t\frac{dx}{dt} + 2t^2x = x(2t + 1)$$

(a) Show that the transformation $x = wt$ transforms this equation into the equation

$$\frac{dw}{dt} + 2wt - 2w = 0$$

(3)

(b) Hence show that the general equation for the displacement of the particle is

$$x = Ate^{2t-t^2}$$

(5)

When the particle has been moving for 2 seconds, it has a displacement of 10 m.

(c) Find the particular solution for the displacement of the particle.

(1)

(d) Find the maximum displacement of the particle, giving your answer to the nearest centimetre.

(4)

(e) Comment on the displacement of the particle in the long term, as predicted by the model, giving a reason for your answer.

(2)

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Question	Scheme	Marks	AOs
8(a)	$x = wt \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{dw}}{\mathrm{d}t}\ t + w$	B1	1.1b
	$t\left(\frac{\mathrm{dw}}{\mathrm{d}t}t + w\right) + 2t^2(wt) = wt(2t+1)$	M1	2.1
	$t^2\frac{\mathrm{dw}}{\mathrm{d}t} + tw + 2t^3w = 2wt^2 + wt$ $t^2\frac{\mathrm{dw}}{\mathrm{d}t} + 2t^3w = 2wt^2$ Leading to $\frac{\mathrm{dw}}{\mathrm{d}t} + 2tw - 2w = 0$ *	A1*	1.1b
		(3)	
(b)	$\int \frac{1}{w}\ dw = 2\int (1-t)\ dt$	M1	3.1a
	$\ln w = 2t - t^2\ (+c)$	M1	1.1b
	Uses correct exponential and ln work to reach $w = \mathrm{e}^{2t-t^2+c} = \mathrm{e}^{2t-t^2}\mathrm{e}^c = A\mathrm{e}^{2t-t^2}$	A1	2.1
	Displacement from O is given by $x = wt = \ldots$	M1	3.4
	$x = At\mathrm{e}^{2t-t^2}$*	A1*	2.2a
		(5)	
(c)	Uses $x = 10$ when $t = 2$ to find the value of A, $10 = 2A$ and achieves $x = 5t\mathrm{e}^{2t-t^2}$	B1	3.4
		(1)	
(d)	Sets $\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \Rightarrow 2t^2x = x(2t+1) \Rightarrow 2t^2 = 2t+1$ or differentiates $x \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 5\mathrm{e}^{2t-t^2} + 5t(2-2t)\mathrm{e}^{2t-t^2} = 0$ to form and solve a quadratic equation.	M1	3.1a
	$t = \frac{1 \pm \sqrt{3}}{2}$	A1	1.1b
	$x = 5\left(\frac{1+\sqrt{3}}{2}\right)\mathrm{e}^{2\left(\frac{1+\sqrt{3}}{2}\right)-\left(\frac{1+\sqrt{3}}{2}\right)^2}$	dM1	1.1b
	Maximum displacement = awrt 16.2 m or 162 cm	A1	3.2a
		(4)	
(e)	$x = 5t\mathrm{e}^{2t-t^2} = 5t\mathrm{e}^{2t}\mathrm{e}^{-t^2}$ or $\frac{5t\mathrm{e}^{2t}}{\mathrm{e}^{t^2}}$	M1	3.4
	As $t \longrightarrow \infty$, $\mathrm{e}^{-t^2} \to 0$ or $\mathrm{e}^{2t-t^2} \to 0$ ∴ displacement from O tends to 0 or $\mathrm{e}^{t^2} \longrightarrow \infty$ ∴ displacement from O tends to 0	A1	2.4
		(2)	
			(15 marks)

Notes:
(a) **B1:** Correct derivative $\frac{\mathrm{d}x}{\mathrm{d}t}$ **M1:** Substitutes in their $\frac{\mathrm{d}x}{\mathrm{d}t}$ **A1*:** Completely correct proof
(b) **M1:** Separates the variables correctly, with d*w* and d*t* the correct positions. **M1:** Integrates both sides to the form $\ln w = \mathrm{f}(t)$ with or without +c **A1:** Uses correct exponential and ln work to reach $w = \mathrm{e}^{2t-t^2+c} = \mathrm{e}^{2t-t^2}\mathrm{e}^{c} = A\mathrm{e}^{2t-t^2}$ Must have + c and a correct intermediate stage. **M1:** Links their equation $w = \mathrm{f}(t)$ to the solution of the model equation correctly. For *x* = *t* 'their *w*' **A1*:** Deduces the correct general equation for the distance
(c) **B1:** Uses *x* = 10 when *t* = 2 to find the correct value of *A*
(d) **M1:** Sets $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ into the differential equation, or uses the product rule to differentiate *x* and sets $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$, to form and solve a quadratic equation. **A1:** Correct value(s) for *t* **dM1:** Dependent of previous method mark, substitutes their value of *t* to find a value for x. **A1:** Maximum displacement = awrt 16.2 m or 162 cm
(e) **M1:** Using the model, separates the exponential terms **A1:** Reason ∴ displacement from *O* tends to 0

6. The concentration of a drug in the bloodstream of a patient, t hours after the drug has been administered, where $t \leqslant 6$, is modelled by the differential equation

$$t^2\frac{\mathrm{d}^2C}{\mathrm{d}t^2} - 5t\frac{\mathrm{d}C}{\mathrm{d}t} + 8C = t^3 \qquad \text{(I)}$$

where C is measured in micrograms per litre.

(a) Show that the transformation $t = \mathrm{e}^x$ transforms equation (I) into the equation

$$\frac{\mathrm{d}^2C}{\mathrm{d}x^2} - 6\frac{\mathrm{d}C}{\mathrm{d}x} + 8C = \mathrm{e}^{3x} \qquad \text{(II)}$$

(5)

(b) Hence find the general solution for the concentration C at time t hours.

(7)

Given that when $t = 6$, $C = 0$ and $\frac{\mathrm{d}C}{\mathrm{d}t} = -36$

(c) find the maximum concentration of the drug in the bloodstream of the patient.

(5)

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Question	Scheme	Marks	AOs
6(a)	**Examples:** $t=\mathrm{e}^x \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}C}=\mathrm{e}^x\frac{\mathrm{d}x}{\mathrm{d}C}$ or $\frac{\mathrm{d}C}{\mathrm{d}x}=t\frac{\mathrm{d}C}{\mathrm{d}t}$ or $\frac{\mathrm{d}C}{\mathrm{d}t}=\mathrm{e}^{-x}\frac{\mathrm{d}C}{\mathrm{d}x}$ or $\frac{\mathrm{d}C}{\mathrm{d}t}=\frac{1}{t}\frac{\mathrm{d}C}{\mathrm{d}x}$	M1	1.1b
	E.g. $\frac{\mathrm{d}C}{\mathrm{d}x}=t\frac{\mathrm{d}C}{\mathrm{d}t}\Rightarrow\frac{\mathrm{d}^2C}{\mathrm{d}x^2}\times\frac{\mathrm{d}x}{\mathrm{d}t}=t\frac{\mathrm{d}^2C}{\mathrm{d}t^2}+\frac{\mathrm{d}C}{\mathrm{d}t}$	**d**M1 A1	2.1 1.1b
	$\frac{\mathrm{d}^2C}{\mathrm{d}x^2}\times\frac{1}{t}=t\frac{\mathrm{d}^2C}{\mathrm{d}t^2}+\frac{1}{t}\frac{\mathrm{d}C}{\mathrm{d}x}\Rightarrow t^2\frac{\mathrm{d}^2C}{\mathrm{d}t^2}=\frac{\mathrm{d}^2C}{\mathrm{d}x^2}-\frac{\mathrm{d}C}{\mathrm{d}x}$ $t^2\frac{\mathrm{d}^2C}{\mathrm{d}t^2}-5t\frac{\mathrm{d}C}{\mathrm{d}t}+8C=\frac{\mathrm{d}^2C}{\mathrm{d}x^2}-\frac{\mathrm{d}C}{\mathrm{d}x}-5\frac{\mathrm{d}C}{\mathrm{d}x}+8C$	**d**M1	2.1
	$\frac{\mathrm{d}^2C}{\mathrm{d}x^2}-6\frac{\mathrm{d}C}{\mathrm{d}x}+8C=\mathrm{e}^{3x}$ *	A1*	1.1b
		(5)	
	<u>Mark (b) and (c) together and ignore labelling</u>		
(b)	$m^2-6m+8=0\Rightarrow m=2,4$	M1	1.1b
	$(C=)\,A\mathrm{e}^{4x}+B\mathrm{e}^{2x}$	A1ft	1.1b
	PI is $C=k\mathrm{e}^{3x}$	B1	2.2a
	$\frac{\mathrm{d}C}{\mathrm{d}x}=3k\mathrm{e}^{3x},\frac{\mathrm{d}^2C}{\mathrm{d}x^2}=9k\mathrm{e}^{3x}\Rightarrow 9k-18k+8k=1\Rightarrow k=-1$	M1	1.1b
	$C=A\mathrm{e}^{4x}+B\mathrm{e}^{2x}-\mathrm{e}^{3x}$	A1	1.1b
	$t=\mathrm{e}^x\Rightarrow C=\ldots$	M1	3.4
	$C=At^4+Bt^2-t^3$	A1	2.2a
		(7)	
(c)	$t=6, C=0\Rightarrow 1296A+36B-216=0$	M1	3.4
	$\frac{\mathrm{d}C}{\mathrm{d}t}=4At^3+2Bt-3t^2\Rightarrow -36=864A+12B-108$	M1	3.4
	$A=0, B=6\Rightarrow C=6t^2-t^3$	A1	1.1b
	$\frac{\mathrm{d}C}{\mathrm{d}t}=12t-3t^2=0\Rightarrow t=4\Rightarrow C=\ldots$	**dd**M1	1.1b
	$C=6(4)^2-(4)^3=32\,\mu\mathrm{g\,L^{-1}}$	A1	3.2a
		(5)	
		(17 marks)	

Notes

(a)

M1: Uses $t = \mathrm{e}^x$ to obtain a correct equation in terms of $\frac{\mathrm{d}C}{\mathrm{d}x}$, $\frac{\mathrm{d}C}{\mathrm{d}t}$ and $t\,(\text{or } \mathrm{e}^x)$ or their reciprocals

dM1: Differentiates again **correctly** with the product rule and chain rule in order to obtain an equation involving $\frac{\mathrm{d}^2C}{\mathrm{d}t^2}$ and $\frac{\mathrm{d}^2C}{\mathrm{d}x^2}$. **This needs to be fully correct calculus work allowing sign errors only.**
A1: Correct equation.
dM1: Shows clearly their substitution into the differential equation (or equivalent work) in order to form the new equation. **Dependent on the first method mark and dependent on having obtained two terms for the second derivative.**
Allow substitution for $\frac{\mathrm{d}C}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2C}{\mathrm{d}x^2}$ into equation (II) to achieve equation (I)
A1*: Fully correct proof with no errors
(b)
M1: Forms and solves a quadratic auxiliary equation $m^2 - 6m + 8 = 0$
A1ft: Correct form for the CF for their AE solutions **which must be distinct and real**
B1: Deduces the correct form for the PI ($k\mathrm{e}^{3x}$)
M1: Differentiates their PI, **which is of the correct form**, and substitutes their derivatives into the DE to find "k"
A1: Correct GS for C in terms of x (**this must be seen explicitly unless implied by subsequent work**)
M1: Links the solution to DE (II) to the solution of the model to find the concentration at time t
A1: Deduces the correct GS for the concentration
If a correct GS is fortuitously found in (b) (e.g. from an incorrect PI form, allow full recovery in (c).
(c)
M1: Uses the conditions of the model ($t = 6$, $C = 0$) to form an equation in A and B.
*******Note that **<u>is</u>** acceptable to use their C in terms of x for this mark as long as they use $x = \ln 6$ when $C = 0$
M1: Uses the conditions of the model $\left(t = 6, \frac{\mathrm{d}C}{\mathrm{d}t} = -36\right)$ to form another equation in A and B.
*******Note that it is **<u>not</u>** acceptable to use $\frac{\mathrm{d}C}{\mathrm{d}x} = -36$ with $x = \ln 6$, as it is necessary to use $\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\mathrm{d}C}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$ e.g. $-36 = \left(4A\mathrm{e}^{4\ln 6} + 2B\mathrm{e}^{2\ln 6} - 3\mathrm{e}^{3\ln 6}\right) \times \mathrm{e}^{-\ln 6}$ or $-216 = 4A\mathrm{e}^{4\ln 6} + 2B\mathrm{e}^{2\ln 6} - 3\mathrm{e}^{3\ln 6}$
A1: Correct equation connecting C with t
ddM1: Uses a suitable method to find the maximum concentration. E.g. solves $\frac{\mathrm{d}C}{\mathrm{d}t} = 0$ for t and substitutes to find C. Allow a solution that solves $\frac{\mathrm{d}C}{\mathrm{d}x} = 0$ for x and uses this correctly to find C.
Dependent on <u>both</u> previous method marks.
A1: Obtains 32 $\mu\mathrm{gL}^{-1}$ using the model. Units are required but allow e.g.

- micrograms per litre
- μg/L
- μg/l
- $\mu\mathrm{g}l^{-1}$

8. A community is concerned about the rising level of pollutant in its local pond and applies a chemical treatment to stop the increase of pollutant.

The concentration, x parts per million (ppm), of the pollutant in the pond water t days after the chemical treatment was applied, is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3 + \cosh t}{3x^2 \cosh t} - \frac{1}{3}x \tanh t \qquad \text{(I)}$$

When the chemical treatment was applied the concentration of pollutant was 3 ppm.

(a) Use the iteration formula

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{(y_{n+1} - y_n)}{h}$$

once to estimate the concentration of the pollutant in the pond water 6 hours after the chemical treatment was applied.

(4)

(b) Show that the transformation $u = x^3$ transforms the differential equation (I) into the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + u \tanh t = 1 + \frac{3}{\cosh t} \qquad \text{(II)}$$

(3)

(c) Determine the general solution of equation (II)

(4)

(d) Hence find an equation for the concentration of pollutant in the pond water t days after the chemical treatment was applied.

(3)

(e) Find the percentage error of the estimate found in part (a) compared to the value predicted by the model, stating if it is an overestimate or an underestimate.

(3)

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Question	Scheme	Marks	AOs
8(a)	At 6 hours $t = 0.25$ so "h" is 0.25	B1	3.1b
	At $t = 0$ $\frac{dx}{dt} = \frac{3+\cosh 0}{3\times 3^2 \cosh 0} - \frac{1}{3}(3)\tanh 0 = ...\left(= \frac{4}{27}\right)$	M1	3.4
	So $x_1 \approx 3 + "0.25" \times "\frac{4}{27}" = ...$	M1	1.1b
	After 6 hours concentration of the pollutant is approximately awrt 3.04 ppm (3 s.f.) or $\frac{82}{27}$ ppm	A1	3.2a
		(4)	
(b)	$\frac{du}{dt} = 3x^2 \times \frac{dx}{dt}$ or $\frac{1}{3x^2}\frac{du}{dt} = \frac{dx}{dt}$ or $\frac{du}{dt} = \frac{du}{dx}\times\frac{dx}{dt} = 3x^2\left(\frac{3+\cosh t}{3x^2\cosh t} - \frac{1}{3}x\tanh t\right)$	B1	2.2a
	So $\frac{dx}{dt} = \frac{3+\cosh t}{3x^2\cosh t} - \frac{1}{3}x\tanh t \to \frac{1}{3x^2}\frac{du}{dt} = \frac{3+\cosh t}{3x^2\cosh t} - \frac{1}{3}x\tanh t$ $\to \frac{du}{dt} = \frac{3}{\cosh t} + 1 - u\tanh t$ $\frac{dx}{dt} = \frac{3+\cosh t}{3x^2\cosh t} - \frac{1}{3}x\tanh t \to \frac{1}{3u^{\frac{2}{3}}}\frac{du}{dt} = \frac{3+\cosh t}{3u^{\frac{2}{3}}\cosh t} - \frac{1}{3}u^{\frac{1}{3}}\tanh t$ $\to \frac{du}{dt} = \frac{3}{\cosh t} + 1 - u\tanh t$	M1	2.1
	$\frac{du}{dt} + u\tanh t = 1 + \frac{3}{\cosh t}$ *	A1*	1.1b
		(3)	
(c)	$\text{I.F.} = \exp\left(\int \tanh t \, dt\right) = \exp(\ln\cosh t) = \cosh t$	B1	2.2a
	$\Rightarrow u'\cosh t' = \int '\cosh t'\left(1 + \frac{3}{\cosh t}\right)dt = \left\{\int \cosh t + 3 \, dt\right\}$	M1	1.1b
	$\Rightarrow u\cosh t = \sinh t + 3t(+c)$	M1	1.1b
	$u\cosh t = \sinh t + 3t + c$ or $u = \tanh t + \frac{3t}{\cosh t} + \frac{c}{\cosh t}$ oe	A1	1.1b
		(4)	
(d)	$t = 0 \Rightarrow x = 3, u = 27 \Rightarrow c = 27\cosh 0 - \sinh 0 - 3(0) = 27$	M1	3.4
	$\Rightarrow x = \left(\tanh t + \frac{3t + "27"}{\cosh t}\right)^{\frac{1}{3}}$	M1	3.4

	$x=\left(\tanh t+\dfrac{3t+27}{\cosh t}\right)^{\frac{1}{3}}$ (oe)	**A1**	3.2a
		(3)	
(e)	$x(0.25)=\left(\tanh 0.25+\dfrac{(0.75+27)}{\cosh 0.25}\right)^{\frac{1}{3}}=...(=3.0055...)$	**M1**	3.4
	% error is $\dfrac{3.0055...-3.037...}{3.0055...}\times 100=...$	**M1**	1.1b
	Estimate in (a) is an overestimate by 1.05% (3 s.f.)	**A1**	3.2a
		(3)	
			(17 marks)

Notes:

(a)

B1: Identifies a correct step length for the situation – 6 hours is a quarter of a day, so h = 0.25

M1: Uses $"y_0"=x(0)=3$ and $t=0$ to find $"\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)_0"=\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)_0$. Accept with whichever notation used, as long as it is clear they are attempting the correct things.

M1: Applies the approximation formula with their "h" and their $"\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)_0"$

A1: For awrt 3.04 ppm. Accept $\dfrac{82}{27}$ ppm

(b)

B1: A correct equation relating $\dfrac{\mathrm{d}u}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ from the chain rule.

M1: Makes a complete substitution for x and $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ in equation (I) or a complete substitution for u and $\dfrac{\mathrm{d}u}{\mathrm{d}t}$ in equation (II)

A1*: Simplifies correctly to achieve the given result.

(c)

B1: Correct integrating factor found or spotted. Allow for $\mathrm{e}^{\ln\cosh t}$

M1: Applies IF to achieve $u"\cosh t"=\int "\cosh t"\left(1+\dfrac{3}{\cosh t}\right)\mathrm{d}t$

M1: A reasonable attempt to integrate the RHS. Need not include constant of integration. If I.F. correct allow for $\pm\sinh t+3t(+c)$

A1: Correct general solution, either implicit or explicit form including the context of integration (award when first seen and isw)

(d)

M1: Uses the initial conditions in an appropriate equation to find the constant of integration. Either t = 0 and u = 27 in the answer to (c), or t = 0 and x = 3 if substitution for x occurs first.

M1: Reverses the substitution and rearranges to find equation for x, with evaluated constant included. **A1:** Correct equation, any equivalent form, but must be $x = \ldots$
(e) **M1:** Uses their model solution to find the value at $t = 0.25$ **M1:** Applies $\frac{\text{actual value} - \text{estimate}}{\text{actual value}} \times 100$ with their values. **A1:** States part (a) is overestimate by 1.05%

9. A particle P moves along a straight line.

At time t minutes, the displacement, x metres, of P from a fixed point O on the line is modelled by the differential equation

$$t^2 \frac{d^2 x}{dt^2} - 2t \frac{dx}{dt} + 2x + 16t^2 x = 4t^3 \sin 2t \qquad \text{(I)}$$

(a) Show that the transformation $x = ty$ transforms equation (I) into the equation

$$\frac{d^2 y}{dt^2} + 16y = 4 \sin 2t$$

(5)

(b) Hence find a general solution for the displacement of P from O at time t minutes.

(8)

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Question	Scheme	Marks	AOs
9(a)	Use of $x = ty$ to give $\frac{dx}{dt} = y + t\frac{dy}{dt}$ or $y = \frac{x}{t} \to \frac{dy}{dt} = -\frac{x}{t^2} + \frac{1}{t}\frac{dx}{dt}$ oe	B1	1.1b
	$\frac{d^2x}{dt^2} = \frac{dy}{dt} + \frac{dy}{dt} + t\frac{d^2y}{dt^2}$ or $\frac{d^2y}{dt^2} = -\frac{1}{t^2}\frac{dx}{dt} + \frac{2x}{t^3} + \frac{1}{t}\frac{d^2x}{dt^2} - \frac{1}{t^2}\frac{dx}{dt}$ oe	M1 A1	2.1 1.1b
	$t^2\left[\text{their } \frac{d^2x}{dt^2}\right] - 2t\left[\text{their } \frac{dx}{dt}\right] + 2[ty] + 16t^2[ty] = 4t^3 \sin 2t$ Or $\left[\text{their } \frac{d^2y}{dt^2}\right] + 16\frac{x}{t} = 4\sin 2t$	dM1	2.1
	$t^2\left[\frac{dy}{dt} + \frac{dy}{dt} + t\frac{d^2y}{dt^2}\right] - 2t\left[y + t\frac{dy}{dt}\right] + 2[ty] + 16t^2[ty] =$ $4t^3 \sin 2t$ $t^3\frac{d^2y}{dt^2} + 16t^3y = 4t^3 \sin 2t \Rightarrow \frac{d^2y}{dt^2} + 16y = 4\sin 2t$ * (oe in reverse)	A1	1.1b
		(5)	
(b)	Solves $m^2 + 16 = 0$ to give $m = \ldots$	M1	1.1b
	$(y =) A\cos 4t + B\sin 4t$	A1	1.1b
	Particular integral $(y =) \underline{\lambda \sin 2t} + \mu \cos 2t$	B1	2.2a
	$\frac{dy}{dt} = 2\lambda\cos 2t - 2\mu\sin 2t$ and $\frac{d^2y}{dt^2} = -4\lambda\sin 2t - 4\mu\cos 2t$	M1	1.1b
	Substitutes into the differential equation and finds values for λ and μ $[-4\lambda\sin 2t - 4\mu\cos 2t] + 16[\lambda\sin 2t + \mu\cos 2t] = 4\sin 2t$ $\Rightarrow \lambda = \ldots,\ \mu = \ldots$	dM1	2.1
	$y = "A\cos 4t + B\sin 4t" + \frac{1}{3}\sin 2t$	A1ft	1.1b
	$x = t[\text{their } y]$	M1	3.4
	$x = t\left[A\cos 4t + B\sin 4t + \frac{1}{3}\sin 2t\right]$	A1	2.2a
		(8)	
			(13 marks)

Notes:

(a)

B1: For a correct suitable first derivative expression linking $\frac{dy}{dt}$ and $\frac{dx}{dt}$.

M1: Uses the product rule to find an equation linking second derivatives from their first derivative expression.

A1: A correct second derivative expression.

dM1: Substitutes the first and second derivatives and replaces x with $ty$to obtain a differential equation in y and t only. Alternatively, may go in reverse and replace y with $\frac{x}{t}$ etc in the second equation to obtain a differential equation in x and t only.

A1*: Simplifies their expression with a correct intermediate stage/working to reach the printed answer. Alternatively, correct working in the other direction to achieve equation (I) from the final equation.

(b)

M1: Forms the correct auxiliary equation and attempts to solve (any values after the correct AE seen)

A1: Correct complementary function. Accept for this mark if they give it in terms of x - you are looking for the correct form for the CF.

B1: Deduces a correct form of the particular integral (must include at least $\lambda \sin 2t$ but may be no more than this). SC if by error the CF includes sin 2t allow B1 for a PI of form $\lambda t \sin 2t + \mu t \cos 2t$

M1: Differentiates the PI twice.

dM1: Dependent on the previous method mark. Substitutes y and $\frac{d^2y}{dt^2}$ into the differential equation leading to values for the constant(s).

A1ft: Correct general equation for y following through their CF, which must be a (non-constant) function of t. Must be in terms of t and start $y = \dots$

M1: Links the solution to the solution of the model equation to find the general solution for the displacement.

A1: Deduces the correct general solution for the displacement. Must be $x = \dots$