Fp1Ch7 XMQs and MS

(Total: 75 marks)

1. FP1_Sample Q2 . 4 marks - FP1ch7 Methods in calculus
2. FP1_Specimen Q7 . 10 marks - FP1ch7 Methods in calculus
3. FP1_2019 Q2 . 4 marks - FP1ch7 Methods in calculus
4. FP1_2019 Q5 . 8 marks - FP1ch7 Methods in calculus
5. FP1_2020 Q1 . 5 marks - FP1ch7 Methods in calculus
6. FP1_2020 Q4 . 8 marks - FP1ch7 Methods in calculus
7. FP1_2020 Q8 . 16 marks - FP1ch7 Methods in calculus
8. FP1_2021 Q2 . 10 marks - FP1ch5 The t-formulae
9. FP1_2022 Q8 . 10 marks - FP1ch6 Taylor series

2. Given k is a constant and that

$$y = x^3 e^{kx}$$

use Leibnitz theorem to show that

$$\frac{d^n y}{dx^n} = k^{n-3}e^{kx}\left(k^3x^3 + 3nk^2x^2 + 3n(n-1)kx + n(n-1)(n-2)\right)$$

(4)

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Question	Scheme	Marks	AOs
2	$y = x^3 e^{kx}$ so $u = x^3$ and $\frac{du}{dx} = 3x^2$ and $\frac{d^2u}{dx^2} = 6x$ and $\frac{d^3u}{dx^3} = 6$ $\left(\text{and } \frac{d^4u}{dx^4} = 0\right)$	M1	1.1b
	$v = e^{kx}$ and $\frac{d^n v}{dx^n} = k^n e^{kx}$ and $\frac{d^{n-1} v}{dx^{n-1}} = k^{n-1} e^{kx}$ and $\frac{d^{n-2} v}{dx^{n-2}} = k^{n-2} e^{kx}$ (and...)	M1	2.1
	$\frac{d^n y}{dx^n} = x^3 k^n e^{kx} + n3x^2 k^{n-1} e^{kx} + \frac{n(n-1)}{2} 6xk^{n-2} e^{kx} + \frac{n(n-1)(n-2)}{3!} 6k^{n-3} e^{kx}$ and remaining terms disappear	M1	2.1
	So $\frac{d^n y}{dx^n} = k^{n-3} e^{kx} \left(k^3 x^3 + 3nk^2 x^2 + 3n(n-1)kx + n(n-1)(n-2)\right)$ *	A1*	1.1b
		(4)	
		(4 marks)	

Notes:

M1: Differentiate $u = x^3$ three times

M1: Use $u = e^{kx}$ and establish the form of the derivatives, with at least the three shown

M1: Uses correct formula, with 2 and 3! (or 6) and with terms shown to disappear after the fourth term

A1*: Correct solution leading to the given answer stated. No errors seen

7. A savings account has a fixed annual interest rate of r% which is compounded n times over the year.

The amount of money in this savings account after one year, £A, is modelled by the equation

$$A = P\left(1 + \frac{r}{100n}\right)^n$$

where £P is the amount in the account at the start of the year.

A student has £1000 in this account at the start of the year and they neither add money to, nor withdraw money from this account.

The annual interest rate is 5%

(a) If the interest is compounded 12 times a year, use the model to show, to the nearest penny, that the amount in the savings account after one year will be £1051.16

(1)

(b) Use L'Hospital's rule to show that $\lim\limits_{n\to\infty}\left(1 + \frac{r}{100n}\right)^n = e^{\frac{r}{100}}$

(6)

(c) Use the model to find the maximum possible amount that the student could have in their savings account after one year.

(3)

Question	Scheme	Marks	AOs
7(a)	$A = 1000\left(1+\frac{5}{1200}\right)^{12} = 1051.16*$	B1*	1.1b
		(1)	
(b)	Let $y = \left(1+\frac{r}{100n}\right)^n$ so $\ln y = \ln\left(1+\frac{r}{100n}\right)^n = n\ln\left(1+\frac{r}{100n}\right)$	M1	3.1a
	$\lim\limits_{n\to\infty} \ln y = \lim\limits_{n\to\infty} n\ln\left(1+\frac{r}{100n}\right) = \lim\limits_{n\to\infty} \frac{\ln\left(1+\frac{r}{100n}\right)}{1/n}$	M1	2.1
	$= \lim\limits_{n\to\infty}\left[\frac{-r/100n^2}{\left(1+\frac{r}{100n}\right)} \div \frac{-1}{n^2}\right]$	dM1 A1	1.1b 1.1b
	$= \lim\limits_{n\to\infty}\frac{r/100}{1+\frac{r}{100n}} = \frac{r}{100}$	A1	1.1b
	$\lim\limits_{n\to\infty}\left(1+\frac{r}{100n}\right)^n = \lim\limits_{n\to\infty} y = \lim\limits_{n\to\infty} e^{\ln y} = e^{\lim\limits_{n\to\infty}\ln y} = e^{\frac{r}{100}}*$	A1*	2.1
		(6)	
(c)	$\lim\limits_{n\to\infty}\left(1+\frac{5}{100n}\right)^n = e^{0.05}$	B1	3.4
	Therefore $\lim\limits_{n\to\infty} 1000\left(1+\frac{5}{100n}\right)^n = 1000e^{0.05}$	M1	2.2a
	Student has £1051.27 in their saving account after one year	A1	3.2a
		(3)	
			(10 marks)

Notes:

(a)

B1*: Using $P = 1000$, $r = 5$ and $n = 12$ to show $A = 1051.16$

(b)

M1: Taking ln's to express $\ln\left(1+\frac{r}{100n}\right)^n = n\ln\left(1+\frac{r}{100n}\right)$

M1: Expressing the limit as a quotient $\lim\limits_{n\to\infty}\ln y = \lim\limits_{n\to\infty}\frac{\ln\left(1+\frac{r}{100n}\right)}{1/n}$

dM1: Applies L'Hospital's rule and attempts to differentiate both the numerator and denominator. Depends on previous method mark

A1: Correct differentiation $\lim\limits_{n\to\infty}\left[\frac{-r/100n^2}{\left(1+\frac{r}{100n}\right)} \div \frac{-1}{n^2}\right]$ simplified or un-simplified

A1: Correct answer for the limit

A1*: Fully correct proof with all mathematical notation cso

(c)

B1: Uses model and the result from part (b) $\lim\limits_{n\to\infty}\left(1+\frac{0.05}{n}\right)^n = e^{0.05}$

M1: Deduces that the amount will be $\lim\limits_{n\to\infty} 1000\left(1+\frac{0.05}{n}\right)^n = 1000e^{0.05}$

A1: Give answer in pounds to 2 decimal places £1051.27

2. Given that k is a real non-zero constant and that

$$y = x^3 \sin kx$$

use Leibnitz's theorem to show that

$$\frac{d^5y}{dx^5} = (k^2x^2 + A)k^3x\cos kx + B(k^2x^2 + C)k^2\sin kx$$

where A, B and C are integers to be determined.

(4)

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Question	Scheme	Marks	AOs
2	$u = x^3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2, \frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 6x, \frac{\mathrm{d}^3u}{\mathrm{d}x^3} = 6$	M1	1.1b
	$v = \sin kx \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = k\cos kx, \frac{\mathrm{d}^2v}{\mathrm{d}x^2} = -k^2\sin kx, \frac{\mathrm{d}^3v}{\mathrm{d}x^3} = -k^3\cos kx,$ $\frac{\mathrm{d}^4v}{\mathrm{d}x^4} = k^4\sin kx, \frac{\mathrm{d}^5v}{\mathrm{d}x^5} = k^5\cos kx$	M1	2.1
	$\frac{\mathrm{d}^5y}{\mathrm{d}x^5} = x^3k^5\cos kx + 5\times3x^2\times k^4\sin kx + \frac{5\times4}{2}\times6x\times\left(-k^3\cos kx\right) +$ $\frac{5\times4\times3}{3!}\times6\times\left(-k^2\sin kx\right)$	M1	2.1
	$= \left(k^2x^2 - 60\right)k^3x\cos kx + 15\left(k^2x^2 - 4\right)k^2\sin kx$	A1	1.1b
		(4)	
		(4 marks)	

Notes

M1: Differentiates $u = x^3$ three times. Need to see $x^3 \to ...x^2 \to ...x \to k$

M1: Uses $v = \sin kx$ to establish the form of the derivatives. Need to see at least alternating $k^{\cdots}\sin kx$ and $k^{\cdots}\cos kx$ with increasing powers of k for at least 3 derivatives.

M1: Uses a correct formula with 2 and 3! (or 6) with terms shown to disappear after the fourth term. This needs to be a correct application of the theorem so that the correct binomial coefficients need to go with the correct pairings of their derivatives. If there is any doubt, at least 3 terms should have the correct structure. Allow equivalent notation for the binomial coefficients e.g. $\binom{5}{0}, \binom{5}{1}$ etc. or ${}^5\mathrm{C}_0, {}^5\mathrm{C}_1$ etc.

A1: Correct expression in the required form with correct values of A, B and C. Apply isw if necessary e.g. if a correct expression is followed by $A = 60$, $B = 15$, $C = -4$
(NB $A = -60$, $B = 15$, $C = -4$)

If there is no use Lebnitz's theorem e.g. repeated differentiation of products, this scores no marks.

5.

$$I = \int \frac{1}{4\cos x - 3\sin x}\,\mathrm{d}x \qquad 0 < x < \frac{\pi}{4}$$

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$I = \frac{1}{5}\ln\left(\frac{2 + \tan\left(\frac{x}{2}\right)}{1 - 2\tan\left(\frac{x}{2}\right)}\right) + k$$

where k is an arbitrary constant.

(8)

Question	Scheme	Marks	AOs
5	$4\cos x - 3\sin x = 4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)$	B1	1.1a
	$\frac{dt}{dx} = \frac{1+t^2}{2}$ or $\frac{dx}{dt} = \frac{2}{1+t^2}$ or $dx = \frac{2dt}{1+t^2}$ or $dt = \frac{1+t^2}{2}dx$ oe	B1 **M1 on ePEN**	2.1
	$\int \frac{1}{4\cos x - 3\sin x}dx = \int \frac{1}{4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$	M1	2.1
	$= \int \frac{2}{4-4t^2-6t}(dt)$ or $\int \frac{1}{2-2t^2-3t}(dt)$ or $\int \frac{-1}{2t^2+3t-2}(dt)$ etc.	A1	1.1b
	$\frac{-2}{4t^2+6t-4} = \frac{-1}{(t+2)(2t-1)} = \frac{A}{(t+2)} + \frac{B}{(2t-1)}$ $\frac{-1}{(t+2)(2t-1)} = \frac{1}{5(t+2)} + \frac{2}{5(1-2t)}$	M1	3.1a
	$\Rightarrow I = \frac{1}{5}\int \frac{1}{(t+2)} - \frac{2}{(2t-1)}(dt)$ or equivalent	A1	1.1b
	$= \frac{1}{5}\int \frac{1}{(t+2)} - \frac{2}{(2t-1)}dt = \frac{1}{5}\ln(t+2) - \frac{1}{5}\ln(1-2t)(+k)$	A1	1.1b
	$= \frac{1}{5}\ln\left(\frac{2+t}{1-2t}\right)(+k) = \frac{1}{5}\ln\left(\frac{2+\tan\left(\frac{x}{2}\right)}{1-2\tan\left(\frac{x}{2}\right)}\right) + k$ *	A1*	2.1
		(8)	
	Alternative for final 4 marks:		
	$= \int \frac{2}{4-4t^2-6t}(dt) = -\frac{1}{2}\int \frac{1}{t^2+\frac{3}{2}t-1}(dt) = -\frac{1}{2}\int \frac{1}{\left(t+\frac{3}{4}\right)^2 - \frac{25}{16}}(dt)$ or e.g. $\int \frac{1}{\frac{25}{8} - 2\left(t+\frac{3}{4}\right)^2}(dt)$	M1 A1	3.1a 1.1b
	$-\frac{1}{2} \times \frac{1}{2} \times \frac{4}{5}\ln\left(\frac{t+\frac{3}{4}-\frac{5}{4}}{t+\frac{3}{4}+\frac{5}{4}}\right)(+c)$	A1	1.1b
	$-\frac{1}{5}\ln\left\|\frac{\tan\left(\frac{x}{2}\right)-\frac{1}{2}}{\tan\left(\frac{x}{2}\right)+2}\right\| + c = \frac{1}{5}\ln\left\|\frac{\tan\left(\frac{x}{2}\right)+2}{\tan\left(\frac{x}{2}\right)-\frac{1}{2}}\right\| + c = \frac{1}{5}\ln\left(\frac{\tan\left(\frac{x}{2}\right)+2}{\frac{1}{2}-\tan\left(\frac{x}{2}\right)}\right) + c$ $= \frac{1}{5}\ln\left(\frac{2\left(\tan\left(\frac{x}{2}\right)+2\right)}{1-2\tan\left(\frac{x}{2}\right)}\right) + c = \frac{1}{5}\ln\left(\frac{\left(\tan\left(\frac{x}{2}\right)+2\right)}{1-2\tan\left(\frac{x}{2}\right)}\right) + \frac{1}{5}\ln 2 + c$ $= \frac{1}{5}\ln\left(\frac{\left(\tan\left(\frac{x}{2}\right)+2\right)}{1-2\tan\left(\frac{x}{2}\right)}\right) + k$	A1*	2.1
		(8 marks)	

Notes
B1: Uses the **correct** formulae to express $4\cos x - 3\sin x$ in terms of t B1(M1 on ePEN): Correct equation in terms of dx, dt and t – can be implied if seen as part of their substitution. M1: Makes a **complete** substitution to obtain an integral in terms of t only. Allow slips with the substitution of "dx" but must be dx = f(t)dt where f(t) ≠ 1. This mark is also available if the candidate makes errors when attempting to simplify $4\left(\frac{1-t^2}{1+t^2}\right)-3\left(\frac{2t}{1+t^2}\right)$ before attempting the substitution. A1: For obtaining a fully correct simplified integral with a constant in the numerator and a 3 term quadratic expression in the denominator. ("dt" not required) M1: Realises the need to express the integrand in terms of partial fractions in order to attempt the integration. **Must have a 3 term quadratic expression in the denominator and a constant in the numerator.** A1: Correct integral in terms of partial fractions – allow any equivalent **correct** integral. ("dt" not required) A1: Fully correct integration in terms of t A1*: Correct solution with no errors including "+ k" (allow "+ c") and with the constant dealt with correctly if necessary. The denominator must also be dealt with correctly. E.g. if it appears as $2t - 1$ initially and becomes $1 - 2t$ without justification, this final mark should be withheld. **<u>Alternative for final 4 marks:</u>** M1: Realises the need to express the integrand in completed square form in order to attempt the integration. **Must have a 3 term quadratic expression in the denominator and a constant in the numerator.** A1: Correct integral with the square completed – allow any equivalent **correct** integral ("dt" not required) A1: Fully correct integration in terms of t A1*: Correct solution with no errors including "+ k" (allow "+ c") and with the constant dealt with correctly if necessary as shown in the scheme and with the denominator dealt with correctly if necessary. Note that it is acceptable for the "dt" to appear and disappear throughout the proof as long as the intention is clear.

1. Use l'Hospital's Rule to show that

$$\lim_{x \to \frac{\pi}{2}} \frac{(e^{\sin x} - \cos(3x) - e)}{\tan(2x)} = -\frac{3}{2}$$

(5)

Question	Scheme	Marks	AOs
1	$\dfrac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\sin x}-\cos(3x)-\mathrm{e}\right)}{\frac{\mathrm{d}}{\mathrm{d}x}(\tan(2x))}=\dfrac{\pm\cos(x)\mathrm{e}^{\sin x}\pm A\sin(3x)}{B\sec^2 2x}$	**M1**	1.1b
	$\dfrac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\sin x}-\cos(3x)-\mathrm{e}\right)}{\frac{\mathrm{d}}{\mathrm{d}x}(\tan(2x))}=\dfrac{\cos(x)\mathrm{e}^{\sin x}+3\sin(3x)}{2\sec^2 2x}$	**A1** **A1**	1.1b 1.1b
	$\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos(x)\mathrm{e}^{\sin x}+3\sin(3x)}{2\sec^2 2x}$ $=\dfrac{\cos\left(\frac{\pi}{2}\right)\mathrm{e}^{\sin\left(\frac{\pi}{2}\right)}+3\sin\left(\frac{3\pi}{2}\right)}{2\sec^2\left(\frac{2\pi}{2}\right)}$ or $=\dfrac{0\times\mathrm{e}+3\times(-1)}{2\times(-1)^2}=\ldots$	**M1**	1.2
	$=-\dfrac{3}{2}$ *	**A1***	2.1
		(5)	
			(5 marks)

Notes:

M1: Attempts differentiation of both numerator and denominator, including at least one use of the chain rule. Either numerator or denominator of the correct form. May be done separately.

A1: Numerator correct

A1: Denominator correct

M1: Applies l'Hospital's Rule, must **see clear use of a substitution** of $x=\dfrac{p}{2}$ into their derivatives, not the original expression. (no need to see check that limits of numerator and denominator are non-zero).

A1*: Needs to be a correct intermediate line following substitution before reaching the printed answer with use of some limit notation. All aspects of the proof should be clear for this mark to be awarded and no errors seen.

4.

$$f(x) = x^4 \sin(2x)$$

Use Leibnitz's theorem to show that the coefficient of $(x - \pi)^8$ in the Taylor series expansion of $f(x)$ about π is

$$\frac{a\pi + b\pi^3}{315}$$

where a and b are integers to be determined.

(8)

$$\left[\begin{array}{c} \textit{The Taylor series expansion of } f(x) \textit{ about } x = k \textit{ is given by} \\ f(x) = f(k) + (x-k)f'(k) + \frac{(x-k)^2}{2!}f''(k) + \ldots + \frac{(x-k)^r}{r!}f^{(r)}(k) + \ldots \end{array}\right]$$

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Question	Scheme	Marks	AOs
	$f(x) = x^4\sin(2x)$ $\quad u = x^4$ $\quad v = \sin(2x)$		
4	$u' = 4x^3, u'' = 12x^2, u''' = 24x, u^{(4)} = 24$ (and $u^{(n)} = 0$ for $n > 4$)	M1	1.1b
	$v' = 2\cos(2x), v'' = -4\sin(2x), v''' = -8\cos(2x), v^{(4)} = 16\sin(2x),$ $v^{(5)} = 32\cos(2x), v^{(6)} = -64\sin(2x), v^{(7)} = -128\cos(2x),$ $v^{(8)} = 256\sin(2x),$	M1 A1 A1	3.1a 1.1b 1.1b
	$f^{(8)}(x) = x^4\times 256\sin(2x) + 8\times 4x^3\times -128\cos(2x)$ Thus $\quad +\frac{8\times 7}{2}\times 12x^2\times -64\sin(2x) + \frac{8\times 7\times 6}{6}\times 24x\times 32\cos(2x)$ $+\frac{8\times 7\times 6\times 5}{24}\times 24\times 16\sin(2x)$ $f^{(8)}(x) = x^4\times 256\sin(2x) + 8\times 4x^3\times -128\cos(2x)$ $+28\times 12x^2\times -64\sin(2x) + 56\times 24x\times 32\cos(2x)$ $+70\times 24\times 16\sin(2x)$	M1	2.1
	$f^{(8)}(\pi) = 0 - 4096\pi^3 - 0 + 1344\times 2^5\pi + 0 \ \left(= -4096\pi^3 + 43008\pi\right)$	M1	1.1b
	Coefficient is $\frac{f^{(8)}(\pi)}{8!} = \frac{1344\times 2^5\pi - 4096\pi^3}{8!\text{ or }\{40320\}}$	M1	2.2a
	$= \frac{336\pi - 32\pi^3}{315}$ (So $a = 336$ and $b = -32$)	A1	2.1
		(8)	
		(8 marks)	

Notes:

M1: Establishes the non-disappearing derivatives of x^4. Allow slips in coefficients, but powers must decrease.

M1: Identifies the relevant derivatives for sin(2x), up to the 8th derivative or establishes the correct pattern. Look for alternating between sin and cos. Condone use of x.

A1: Correct sizes for the coefficients, allow sign errors for this mark (may be due to incorrect signs when differentiating sin and cos) Must have angle 2x.

A1: All derivatives correctly established. (Note the sin terms may be omitted if the student has made clear they will disappear, but if present they must be correct).

M1: Applies Leibnitz's theorem to get the 8th derivative with their expressions. Binomial coefficients must be present.

M1: Evaluates their 8th derivative at π

M1: Uses Taylor series – divides their value for $f^{(8)}(\pi)$ by 8!

A1: Simplifies to the correct answer.

<u>Note:</u> If do not use Leibnitz's theorem then maximum M0 M0 A0 A0 M0 M1 M1 A0

8.

$$f(x) = \frac{3}{13 + 6\sin x - 5\cos x}$$

Using the substitution $t = \tan\left(\frac{x}{2}\right)$

(a) show that $f(x)$ can be written in the form

$$\frac{3(1+t^2)}{2(3t+1)^2 + 6}$$

(3)

(b) Hence solve, for $0 < x < 2\pi$, the equation

$$f(x) = \frac{3}{7}$$

giving your answers to 2 decimal places where appropriate.

(5)

(c) Use the result of part (a) to show that

$$\int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} f(x)\,dx = K\left(\arctan\left(\frac{\sqrt{3}-9}{3}\right) - \arctan\left(\frac{\sqrt{3}+3}{3}\right) + \pi\right)$$

where K is a constant to be determined.

(8)

Question	Scheme		Marks	AOs
8(a)	$\text{f}(x) = \dfrac{3}{13+6\times\dfrac{2t}{1+t^2}-5\times\dfrac{1-t^2}{1+t^2}}$		**M1**	1.1b
	$= \dfrac{3(1+t^2)}{13(1+t^2)+12t-5(1-t^2)}$		**M1**	1.1b
	$= \dfrac{3(1+t^2)}{18t^2+12t+8} \Rightarrow$ for example $\dfrac{3(1+t^2)}{2(9t^2+6t+1)+6}$ or $\dfrac{3(1+t^2)}{2\left[(3t+1)^2-1\right]+8}$ $\Rightarrow \dfrac{3(1+t^2)}{2(3t+1)^2+6}$ *		**A1***	2.1
			(3)	
(b)	$\text{f}(x) = \dfrac{3}{7} \Rightarrow \dfrac{3(1+t^2)}{2(3t+1)^2+6} = \dfrac{3}{7} \Rightarrow 21+21t^2 = 54t^2+36t+24$ $\Rightarrow 11t^2+12t+1=0$		**M1**	1.1b
	$\Rightarrow (11t+1)(t+1)=0 \Rightarrow t = ...$		**M1**	1.1b
	$t=-1, t=-\dfrac{1}{11}$		**A1**	1.1b
	$\Rightarrow x = 2\arctan\left(\text{"}\textit{their } t\text{"}\right)+2\pi$ for a negative t		**dM1**	3.1a
	$x=\dfrac{3\pi}{2}$ or awrt 4.71 and awrt $x=6.10$		**A1**	1.1b
			(5)	
(c)	$\displaystyle\int \text{f}(x) = \int \frac{3(1+t^2)}{2(3t+1)^2+6}\times\frac{2}{1+t^2}\,\text{d}t = \int \frac{3}{(3t+1)^2+3}\,\text{d}t$		**B1**	2.1
	$= K\arctan\left(M(3t+1)\right)$	$u=(3t+1) \Rightarrow K\arctan\left(Mu\right)$	**M1**	1.1b
	$= \dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{3t+1}{\sqrt{3}}\right)$	$= \dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{u}{\sqrt{3}}\right)$	**A1**	1.1b
	$\displaystyle\int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} \text{f}(x)\,\text{d}x = \int_{\frac{\pi}{3}}^{\pi} \text{f}(x)\,\text{d}x + \int_{\pi}^{\frac{4\pi}{3}} \text{f}(x)\,\text{d}x$ $\displaystyle= \int_{\frac{\sqrt{3}}{3}}^{\infty} ...\,\text{d}t + \int_{-\infty}^{-\sqrt{3}} ...\,\text{d}t$ or $\displaystyle\int_{\sqrt{3}+1}^{\infty} ...\,\text{d}u + \int_{-\infty}^{1-3\sqrt{3}} ...\,\text{d}u$		**B1**	3.1a
	$= \dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{3\left(-\sqrt{3}\right)+1}{\sqrt{3}}\right) - \dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{3\left(\dfrac{\sqrt{3}}{3}\right)+1}{\sqrt{3}}\right) + ...$		**M1**	1.1b

	$=\frac{\sqrt{3}}{3}\left(\arctan\left(\frac{\sqrt{3}-9}{3}\right)-\arctan\left(\frac{\sqrt{3}+3}{3}\right)\right)+\ldots$	**A1**	1.1b
	$=\ldots+\lim_{t\to\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{3t+1}{\sqrt{3}}\right)-\lim_{t\to-\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{3t+1}{\sqrt{3}}\right)=\ldots+\frac{\pi}{2\sqrt{3}}-\left(-\frac{\pi}{2\sqrt{3}}\right)$	**M1**	3.1a
	$=\frac{\sqrt{3}}{3}\left(\arctan\left(\frac{\sqrt{3}-9}{3}\right)-\arctan\left(\frac{\sqrt{3}+3}{3}\right)+\pi\right)$	**A1**	2.1
		(8)	
(16 marks)			

Notes:

(a)

M1: Uses one correct substitution

M1: Both substitutions correct and attempts to multiply through numerator and denominator by $1+t^2$.

A1*: Completes to the correct expression with no errors seen. Must see an intermediate step simplifying the denominator – most likely one of the ones seen in the scheme.

(b)

M1: Equates the result in (a) to $\frac{3}{7}$ and simplifies to a 3TQ

M1: Solves their equation by any valid means.

A1: Correct values for t

dM1: Dependent on first method mark. Applies the correct process to find at least one value for x from a negative value for t. (If two positive values are found in error, this mark cannot be scored.)

A1: Both answers correct and no others in range.

(c)

B1: Applies the substitution including the use of $\mathrm{d}x=\frac{2}{1+t^2}\mathrm{d}t$

M1: Attempts the integration to achieve $K\arctan\left(M(1+3t)\right)$ or $K\arctan\left(Mu\right)$ if using a substitution of $u=(3t+1)$.

May use substitution $3t+1=\sqrt{3}\tan\theta \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}\theta}=\frac{\sqrt{3}}{3}\sec^2\theta \Rightarrow \int\frac{\sqrt{3}}{3}\mathrm{d}\theta=\frac{\sqrt{3}}{3}\theta=\frac{\sqrt{3}}{3}\arctan\left(\frac{3t+1}{\sqrt{3}}\right)$

A1: Correct integral.

B1: Changes the limits and splits the integral around π

M1: Applies their limits '$1/\sqrt{3}$' and '$-\sqrt{3}$' to their integrand.

A1: Correct "arctan" expressions.

M1: Correct work to evaluate the $\pm\infty$ limits

A1: Fully correct solution.

2. (i) Use the substitution $t = \tan\dfrac{x}{2}$ to prove the identity

$$\frac{\sin x - \cos x + 1}{\sin x + \cos x - 1} \equiv \sec x + \tan x \qquad x \neq \frac{n\pi}{2} \qquad n \in \mathbb{Z}$$

(5)

(ii) Use the substitution $t = \tan\dfrac{\theta}{2}$ to determine the exact value of

$$\int_0^{\frac{\pi}{2}} \frac{5}{4 + 2\cos\theta}\, d\theta$$

giving your answer in simplest form.

(5)

Question	Scheme	Marks	AOs
2(i)	$\dfrac{\sin x-\cos x+1}{\sin x+\cos x-1}=\dfrac{\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}+1}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}-1}=\ldots$	**M1**	1.1b
	$=\dfrac{2t-(1-t^2)+1+t^2}{2t+1-t^2-(1+t^2)}$ or $=\dfrac{2t-1+t^2+1+t^2}{2t+1-t^2-1-t^2)}$ numerator $=\dfrac{2t-(1-t^2)+1+t^2}{1+t^2}$ denominator = $=\dfrac{2t+1-t^2-(1+t^2)}{1+t^2}$ and divides	**M1**	2.1
	$=\dfrac{2t^2+2t}{2t-2t^2}\left(=\dfrac{1+t}{1-t}\right)$	**A1**	1.1b
	$=\dfrac{t+1}{1-t}\times\dfrac{1+t}{1+t}=\dfrac{t^2+2t+1}{1-t^2}=\dfrac{1+t^2}{1-t^2}+\dfrac{2t}{1-t^2}$ Alt: $\sec x+\tan x=\dfrac{1+t^2}{1-t^2}+\dfrac{2t}{1-t^2}=\dfrac{1+2t+t^2}{1-t^2}$	**M1**	3.1a
	$=\dfrac{1}{\cos x}+\tan x=\sec x+\tan x$ * Alt: $=\dfrac{(t+1)^2}{(1-t)(1+t)}=\dfrac{1+t}{1-t}=LHS$ hence result proved.*	**A1***	2.1
		(5)	
(ii)	$\displaystyle\int_{(0)}^{\left(\frac{\pi}{2}\right)}\frac{5}{4+2\cos\theta}\,\mathrm{d}\theta=\int_{(0)}^{(1)}\frac{5}{4+2\frac{1-t^2}{1+t^2}}\times\frac{2}{1+t^2}\,\mathrm{d}t$ **Alternatively** $\dfrac{\mathrm{d}t}{\mathrm{d}\theta}=\dfrac{1}{2}\sec^2\left(\dfrac{\theta}{2}\right)=\dfrac{1}{\cos\theta+1}$ leading to $\displaystyle\int_{(0)}^{\left(\frac{\pi}{2}\right)}\frac{5}{4+2\cos\theta}\,\mathrm{d}\theta=\int_{(0)}^{(1)}\frac{5\cos\theta+5}{4+2\cos\theta}\,\mathrm{d}t=\int_{(0)}^{(1)}\frac{5\left(\frac{1-t^2}{1+t^2}\right)+5}{4+2\left(\frac{1-t^2}{1+t^2}\right)}\,\mathrm{d}t$	**M1**	2.1
	$\displaystyle=\int_{(0)}^{(1)}\frac{10}{4(1+t^2)+2(1-t^2)}\,\mathrm{d}t=\int_{(0)}^{(1)}\frac{5}{3+t^2}\,\mathrm{d}t$ o.e.	**A1**	1.1b
	$=\left[5\times\dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{t}{\sqrt{3}}\right)\right]_{(0)}^{(1)}$	**M1**	1.1b

	$=\frac{5}{\sqrt{3}}\left(\arctan\left(\frac{1}{\sqrt{3}}\right)-0\right)$ or $\frac{5}{\sqrt{3}}\left(\arctan\left(\frac{\tan\left(\frac{\pi}{4}\right)}{\sqrt{3}}\right)-\arctan\left(\frac{\tan(0)}{\sqrt{3}}\right)\right)$	**M1**	2.2a
	$=\frac{5\pi\sqrt{3}}{18}$ oe in a surd form e.g. $\frac{5\pi}{6\sqrt{3}}$	**A1**	1.1b
		(5)	
(10 marks)			

Notes:

(i)

M1: Applies the *t*-formulae to the left-hand side of expression. Allow slips in signs of the terms.

M1: Multiplies numerator and denominator through by $1+t^2$ (allow if they forget to multiply the 1's). Alternative works separately on the numerator and denominator to combine terms and then divides.

A1: Correct $\frac{\text{quadratic}}{\text{quadratic}}$ with terms gathered, award where first seen, need not have cancelled 2*t* for this mark.

M1: Cancels 2*t*, multiplies numerator and denominator by $1+t$ and splits to sum of two terms. If working from both sides, this mark is for substituting the *t*-formulae into the right-hand side and combining to single fraction.

A1*: Correct completion to given result. No errors in proof. If working from both sides, a suitable conclusion is needed, e.g "hence proven".

(ii)

M1: Applies the substitution including the use of $\mathrm{d}\theta=\frac{2}{1+t^2}\mathrm{d}t$ (Limits not needed for first three marks).

A1: Simplifies correctly to a recognisable integrable form.

M1: Integrates to the form $K\arctan\left(\frac{t}{a}\right)$ where a^2 is their constant term.

M1: Deduces correct limits and applies them the correct way round OR deduces integral in terms of θ from their integration and applies original limits the correct way.

A1: $\frac{5\pi\sqrt{3}}{18}$ or equivalent in surd form.

Note use of calculator does not lead to the exact value required in the question = 1.51149947

This can score M1 A1 M0 M1 A0

8. [*The Taylor series expansion of* f(*x*) *about* *x* = *a* *is given by*

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \ldots$$]

(i) (a) Use differentiation to determine the Taylor series expansion of $\ln x$, in ascending powers of $(x-1)$, up to and including the term in $(x-1)^2$

(4)

(b) Hence prove that

$$\lim_{x\to 1}\left(\frac{\ln x}{x-1}\right) = 1$$

(2)

(ii) Use L'Hospital's rule to determine

$$\lim_{x\to 0}\left(\frac{1}{(x+3)\tan(6x)\operatorname{cosec}(2x)}\right)$$

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

Question	Scheme	Marks	AOs
8(i) (a)	$f(x) = ln\,x \quad \Rightarrow f(1) = 0$ $f'(x) = \frac{1}{x} \quad \Rightarrow f'(1) = 1$ $f^{''\frac{1}{x^2}f''}$	M1 A1	1.1b 1.1b
	$(ln\,x) = (0+)(x-1) - \frac{1}{2}(x-1)^2 + \ldots$	M1 A1	2.5 1.1b
		(4)	
(i) (b)	$\lim_{x \to 1}\left(\frac{ln\,x}{x-1}\right) = \lim_{x \to 1}\left(\frac{(x-1) - \frac{1}{2}(x-1)^2 + \ldots}{x-1}\right)$ $= \lim_{x \to 1}\left(1 - \frac{1}{2}(x-1) + \ldots\right)$	M1	2.1
	$= \lim_{x \to 1}\left(1 - \frac{1}{2}(x-1) + \ldots\right) = 1$* cso	A1*	2.2a
		(2)	
(ii)	Writes as an indeterminate form For example $\frac{sin(2x)}{(x+3)\,tan(6x)}$ or $\frac{sin(2x)\,cos(6x)}{(x+3)\,sin(6x)}$	M1	3.1a
	Differentiates numerator and denominator using appropriate rules $\frac{2\,cos(2x)}{tan(6x) + 6(x+3)\,sec^2(6x)}$ or $\frac{2\,cos(2x)\,cos(6x) - 6\,sin(2x)\,sin(6x)}{sin(6x) + 6(x+3)\,cos(6x)}$	M1 A1	1.1b 1.1b
	$\lim_{x \to 0}\left(\frac{1}{(x+3)\,tan(6x)\text{cosec}(2x)}\right) = \frac{2}{18} = \frac{1}{9}$ o.e	A1cso	2.2a
		(4)	
		(10 marks)	

Notes:

(i) (a) Notes: ignore extra terms throughout.

M1: Differentiates $f(x) = ln\,x$ twice and finds $f(1)$, f$'(1)$ and f$''(1)$

A1: Correct differentiation and values for $f(1)$, f$'(1)$ and f$''(1)$.

M1: Uses correct mathematical notation to find the Taylor series for $ln\,x$ in powers of $(x-1)$ up to $(x-1)^2$

A1: Correct expansion with simplified coefficients. Do not be concerned with the left hand side.

(i) (b) Question says "hence" so the result of (a) must be used. No marks for l'Hosptial's rule on the original functions (send to review if attempted with their part (a)).

M1: Substitutes their Taylor series for $ln\,x$ in powers of $(x-1)$ up to $(x-1)^2$ into the limit and cancels a factor $(x-1)$ from each term. Allow for the cancelling seeing a relevant strikethrough in all $x-1$ terms.

A1*: $\lim_{x \to 1}\left(\frac{ln\,x}{x-1}\right) = \lim_{x \to 1}\left(1 - \frac{1}{2}(x-1) + \ldots\right) = 1$ cso Must have come from a correct expansion. Must see the $1 - \frac{1}{2}(x-1)$

(ii)

M1: Writes the fraction in an indeterminate form $\frac{f(x)}{g(x)}$ where $\frac{\boldsymbol{f}(0)}{\boldsymbol{g}(0)} = \frac{0}{0}$ or $\frac{\boldsymbol{f}(0)}{\boldsymbol{g}(0)} = \frac{"\infty"}{"\infty"}$

M1: Differentiates numerator and denominator using appropriate rules, ie product rule for a product etc. Allow slips in coefficients but the form should be correct. This mark is available as long as written as $\frac{f(x)}{g(x)}$ even if not an indeterminate form. May be seen written as separate from the fraction, ie f'(x) = … and g'(x) = …

A1: Depend on both Ms. Correct differentiation for derivatives that lead to a limit. Must have a derivative for which g'(0) ≠ 0 and is finite.

A1cso: Deduces the correct limit from fully correct work.