Fp1Ch6 XMQs and MS

(Total: 30 marks)

1. FP1_Sample   Q4 . 9 marks - FP1ch6 Taylor series
2. FP1_2019   Q3 . 9 marks - FP1ch6 Taylor series
3. FP1_2021   Q6 . 12 marks - FP1ch6 Taylor series

4.

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y = 0 \qquad \text{(I)}$$

(a) Show that

$$\frac{d^5y}{dx^5} = ax\frac{d^4y}{dx^4} + b\frac{d^3y}{dx^3}$$

where a and b are integers to be found.

(4)

(b) Hence find a series solution, in ascending powers of x, as far as the term in x^5, of the differential equation (I) where $y = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$

(5)

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Question	Scheme	Marks	AOs
4(a)	$y''=2xy'-y \Rightarrow y'''=2xy''+2y'-y'$	M1	1.1b
		A1	1.1b
	$y'''=2xy''+y' \Rightarrow y''''=2xy'''+2y''+y''$	M1	2.1
	$y''''=2xy'''+3y'' \Rightarrow y'''''=2xy''''+5y'''$	A1	2.1
		(4)	
(b)	$x=0, y=0, y'=1 \Rightarrow y''(0)=0\pi$ from equation	B1	2.2a
	$y'''(0)=2\times0\times y''(0)+1=1$; $y''''(0)=2\times0\times1+3\times0=0$;	M1	1.1b
	$x=0,\ y'''(0)=1,\ y''''(0)=0 \Rightarrow y'''''(0)=5$	A1	1.1b
	$y=y(0)+y'(0)x+\frac{y''(0)}{2}x^2+\frac{y'''(0)}{6}x^3+\frac{y''''(0)}{24}x^4+\frac{y'''''(0)}{120}x^5+...$	M1	2.5
	Series solution: $y=x+\frac{1}{6}x^3+\frac{1}{24}x^5+...$	A1ft	1.1b
		(5)	
			(9 marks)

Notes:

(a)

M1: Attempts to differentiate equation with use of the product rule

A1: cao. Accept if terms all on one side

M1: Continues the process of differentiating to progress towards the goal. Terms may be kept on one side, but an expression in the fourth derivative should be obtained

A1: Completes the process to reach the fifth derivative and rearranges to the correct form to obtain the correct answer by correct solution only

(b)

B1: Deduces the correct value for $y''(0)$ from the information in the question

M1: Finds the values of the derivatives at the given point

A1: All correct

M1: Correct mathematical language required with given denominators. Can be in factorial form

A1ft: Correct series, must start $y=$ …. Follow through the values of their derivatives at 0

3.

$$\frac{dy}{dx} = x - y^2 \qquad \text{(I)}$$

(a) Show that

$$\frac{d^5y}{dx^5} = ay\frac{d^4y}{dx^4} + b\frac{dy}{dx}\frac{d^3y}{dx^3} + c\left(\frac{d^2y}{dx^2}\right)^2$$

where a, b and c are integers to be determined.

(4)

(b) Hence find a series solution, in ascending powers of x as far as the term in x^5, of the differential equation (I), given that $y = 1$ at $x = 0$

(5)

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Question	Scheme	Marks	AOs
3(a)	$\frac{d^2y}{dx^2}=1-2y\frac{dy}{dx}\Rightarrow\frac{d^3y}{dx^3}=-2y\frac{d^2y}{dx^2}-2\left(\frac{dy}{dx}\right)^2$	M1 A1	1.1b 1.1b
	$\frac{d^4y}{dx^4}=-2\frac{dy}{dx}\frac{d^2y}{dx^2}-2y\frac{d^3y}{dx^3}-4\frac{dy}{dx}\frac{d^2y}{dx^2}=-6\frac{dy}{dx}\frac{d^2y}{dx^2}-2y\frac{d^3y}{dx^3}$	**d**M1	2.1
	$\frac{d^5y}{dx^5}=-6\frac{dy}{dx}\frac{d^3y}{dx^3}-6\left(\frac{d^2y}{dx^2}\right)^2-2y\frac{d^4y}{dx^4}-2\frac{dy}{dx}\frac{d^3y}{dx^3}$ $=-2y\frac{d^4y}{dx^4}-8\frac{dy}{dx}\frac{d^3y}{dx^3}-6\left(\frac{d^2y}{dx^2}\right)^2$	A1	2.1
		(4)	
(b)	$x=0, y=1\Rightarrow\left(\frac{dy}{dx}\right)_0=0-1^2=-1$	B1	2.2a
	$\left(\frac{d^2y}{dx^2}\right)_0=1-2(1)(-1)=3, \left(\frac{d^3y}{dx^3}\right)_0=-2(1)(3)-2(-1)^2=-8$ $\left(\frac{d^4y}{dx^4}\right)_0=-6(-1)(3)-2(1)(-8)=34,$ $\left(\frac{d^5y}{dx^5}\right)_0=-2(1)(34)-8(-1)(-8)-6(3)^2=-186$	M1 A1	1.1b 1.1b
	$y=y(0)+x\left(\frac{dy}{dx}\right)_0x+\frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0+\frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0+\frac{x^4}{4!}\left(\frac{d^4y}{dx^4}\right)_0+\frac{x^5}{5!}\left(\frac{d^5y}{dx^5}\right)_0+...$ With their values	M1	2.5
	$(y=)1-x+\frac{3}{2}x^2-\frac{8}{6}x^3+\frac{34}{24}x^4-\frac{186}{120}x^5+...$ $(y=)1-x+\frac{3}{2}x^2-\frac{4}{3}x^3+\frac{17}{12}x^4-\frac{31}{20}x^5+...$	A1ft	1.1b
		(5)	
		(9 marks)	

Notes

(a)

M1: Attempts to find the second and third derivatives:

This requires $\frac{d^2y}{dx^2}=1\pm2y\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}=\pm2y\frac{dy}{dx}$ followed by $\frac{d^3y}{dx^3}=\pm2y\frac{d^2y}{dx^2}\pm...$ or $\frac{d^3y}{dx^3}=\pm...\pm2\left(\frac{dy}{dx}\right)^2$

A1: Correct second **and** third derivatives.

dM1: Continues to differentiate to reach the 5th derivative. **This is dependent on the first method mark but there is no need to check the detail and the mark can be awarded as long as the 5th derivative is reached.**

A1: Completes the process, collecting terms if necessary, to obtain the correct expression (NB $a = -2$, $b = -8$, $c = -6$)

Allow dash/dot notation for the derivatives but the final answer must be in the correct form.

Note that if $\frac{d^2y}{dx^2} = \pm 2y\frac{dy}{dx}$ is obtained initially, allow a full recovery in (a).

Note that (a) can be found using Leibnitz's theorem and the following scheme should be applied:

M1: $\frac{d^2y}{dx^2} = 1 \pm 2y\frac{dy}{dx}$ or $\frac{d^2y}{dx^2} = \pm 2y\frac{dy}{dx}$ followed by an attempt to differentiate y 3 times and $\frac{dy}{dx}$ 3 times.

A1: All correct

dM1: $\frac{d^5y}{dx^5} = -2\frac{dy}{dx}\frac{d^4y}{dx^4} - 3\times 2\frac{dy}{dx}\frac{d^3y}{dx^3} - 3\times 2\left(\frac{d^2y}{dx^2}\right)^2 - 2y\frac{d^3y}{dx^3}\frac{dy}{dx}$ (correct application of Leibnitz)

A1: $= -2y\frac{d^4y}{dx^4} - 8\frac{dy}{dx}\frac{d^3y}{dx^3} - 6\left(\frac{d^2y}{dx^2}\right)^2$

As in the main scheme, if $\frac{d^2y}{dx^2} = \pm 2y\frac{dy}{dx}$ is obtained initially, allow a full recovery in (a).

Alternative for (a):

M1: $\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx} = 1 - 2y\left(x - y^2\right) = 1 - 2xy + 2y^3 \Rightarrow \frac{d^3y}{dx^3} = -2y - 2x\frac{dy}{dx} + 6y^2\frac{dy}{dx}$

Score for the second derivative form as in the main scheme and then an attempt at the third derivative with at least 2 terms correct.

A1: Fully correct

Then as main scheme.

(b)

B1: Deduces the correct value for $y'(0)$

M1: Finds the values of all the other derivatives at $x = 0$ up to 5th. There is no need to check their values as long as there is no obvious incorrect work, but values for all the derivatives up to the 5th must be found

A1: All values correct (as single values – e.g. do not allow unsimplified)

M1: Applies the correct Maclaurin series for their values including the factorials up to the term in x^5

A1ft: Correct expansion, follow through their values for the derivatives. This does not have to be simplified but the factorials need to be evaluated. **Once a correct, or correct follow through expression is seen apply isw.**

6.

The Taylor series expansion of $f(x)$ *about* $x = a$ *is given by*

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \ldots$$

Given that

$$y = (1 + \ln x)^2 \qquad x > 0$$

(a) show that $\dfrac{d^2y}{dx^2} = -\dfrac{2\ln x}{x^2}$

(4)

(b) Hence find $\dfrac{d^3y}{dx^3}$

(2)

(c) Determine the Taylor series expansion about $x = 1$ of

$$(1 + \ln x)^2$$

in ascending powers of $(x - 1)$, up to and including the term in $(x - 1)^3$

Give each coefficient in simplest form.

(3)

(d) Use this series expansion to evaluate

$$\lim_{x \to 1} \frac{2x - 1 - (1 + \ln x)^2}{(x-1)^3}$$

explaining your reasoning clearly.

(3)

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Question	Scheme	Marks	AOs
6(a)	$y=(1+\ln x)^2 \Rightarrow \frac{dy}{dx}=k(1+\ln x)\times\frac{1}{x}$ or $y=1+2\ln x+(\ln x)^2 \Rightarrow \frac{dy}{dx}=\frac{A}{x}+B\ln x\times\frac{1}{x}$	**M1**	1.1b
	$y=(1+\ln x)^2 \Rightarrow \frac{dy}{dx}=2(1+\ln x)\times\frac{1}{x}$ or $y=1+2\ln x+(\ln x)^2 \Rightarrow \frac{dy}{dx}=\frac{2}{x}+2\ln x\times\frac{1}{x}$	**A1**	1.1b
	$\frac{d^2y}{dx^2}=\frac{k\left(\frac{1}{x}\right)\times x-k(1+\ln x)\times 1}{x^2}$ or $\frac{k}{x}x^{-1}+k(1+\ln x)(-x^{-2})$ or $\frac{d^2y}{dx^2}=-\frac{A}{x^2}+\frac{\frac{B}{x}\times x-2\ln x\times 1}{x^2}$	**M1**	1.1b
	$\frac{d^2y}{dx^2}=\frac{\left(\frac{2}{x}\right)\times x-2(1+\ln x)}{x^2}$ or $\frac{2}{x}x^{-1}+2(1+\ln x)(-x^{-2})$ Leading to $\frac{d^2y}{dx^2}=-\frac{2\ln x}{x^2}$ * achieved from correct work.	**A1***	2.1
		(4)	
(b)	$\frac{d^3y}{dx^3}=\frac{\pm\frac{C}{x}\times x^2\pm Dx\ln x}{x^4}$ or $\frac{d^3y}{dx^3}=(-2\ln x)(-Cx^{-3})+\left(-\frac{D}{x}\right)(x^{-2})$	**M1**	1.1b
	$\frac{d^3y}{dx^3}=-\frac{\frac{2}{x}\times x^2-4x\ln x}{x^4}$ or $\frac{-\frac{2}{x}\times x^2-(-2\ln x)(2x)}{x^4}$ or $-\frac{2}{x^3}+\frac{4\ln x}{x^3}$	**A1**	1.1b
		(2)	
(c)	$y(1)=1, y'(1)=2, y''(1)=0, y'''(1)=-2$	**M1**	1.1b
	$[y]=1+2(x-1)+\frac{0}{2!}(x-1)^2+\frac{-2}{3!}(x-1)^3+\ldots$	**M1**	2.5
	$[y]=1+2(x-1)-\frac{1}{3}(x-1)^3+\ldots$ or $[y]=-1+2x-\frac{1}{3}(x-1)^3+\ldots$	**A1**	1.1b
		(3)	
(d)	$\frac{2x-1-(1+\ln x)^2}{(x-1)^3}=\frac{2x-1-1-2(x-1)+\frac{1}{3}(x-1)^3+\ldots}{(x-1)^3}=\frac{\frac{1}{3}(x-1)^3+\ldots}{(x-1)^3}$	**M1**	1.1b

	Simplifies and realises that terms cancel to leave a constant term $\dfrac{\frac{1}{3}\cancel{(x-1)^3}+\ldots}{\cancel{(x-1)^3}}=\dfrac{1}{3}$	**M1**	3.1a
	Hence $\lim_{x\to 1}\dfrac{2x-1-(1+\ln x)^2}{(x-1)^3}=\dfrac{1}{3}$ as all remaining terms will become zero in the limit as they are multiples of $(x-1)^k$, which tends to 0.	**A1**	2.4
		(3)	
(12 marks)			

Notes:

(a)

M1: Attempts the first derivative, including use of the chain rule. May expand first. E.g. accept forms as shown.

A1: Correct first derivative, need not be simplified.

M1: Attempts second derivative using quotient rule or product rule – examples as shown, or equivalents accepted.

A1*: Correct result achieved from correct work.

(b)

M1: Applies quotient rule or product rule to achieve third derivative. If formula is quoted it must be correct, if not accept derivatives of the form shown as there may be confusion with the minus sign.

A1: Correct third derivative, any form.

(c)

M1: Find value of derivatives at $x = 1$.

M1: Applies Taylor series expansion

A1: Correct series, may be unsimplified, isw once correct series seen. Must be using a correct third derivative.

(d)

M1: Applies the series to the limit and cancels terms in numerator to leave term in $(x-1)^3$ and above only (may not see +… for this mark)

M1: Simplifies and realises that the $x-1^{\ 3}$ cancels and achieves a constant A

A1: Correct limit deduced with reasoning given why the remaining terms disappear.