# Fp1Ch5 XMQs and MS

(Total: 72 marks)

1. FP1_Sample   Q8 . 15 marks - FP1ch5 The t-formulae
2. FP1_Specimen Q3 .  7 marks - FP1ch5 The t-formulae
3. FP1_2022     Q2 .  7 marks - FP1ch5 The t-formulae
4. FP1(AS)_2018 Q1 .  7 marks - FP1ch5 The t-formulae
5. FP1(AS)_2019 Q1 .  9 marks - FP1ch5 The t-formulae
6. FP1(AS)_2020 Q3 . 11 marks - FP1ch5 The t-formulae
7. FP1(AS)_2021 Q3 .  9 marks - FP1ch5 The t-formulae
8. FP1(AS)_2022 Q3 .  7 marks - FP1ch5 The t-formulae

**8.**

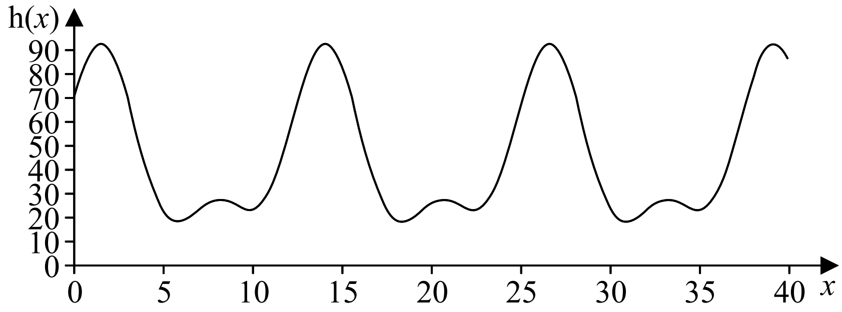



**Figure 1**

Figure 1 shows the graph of the function h($x$) with equation

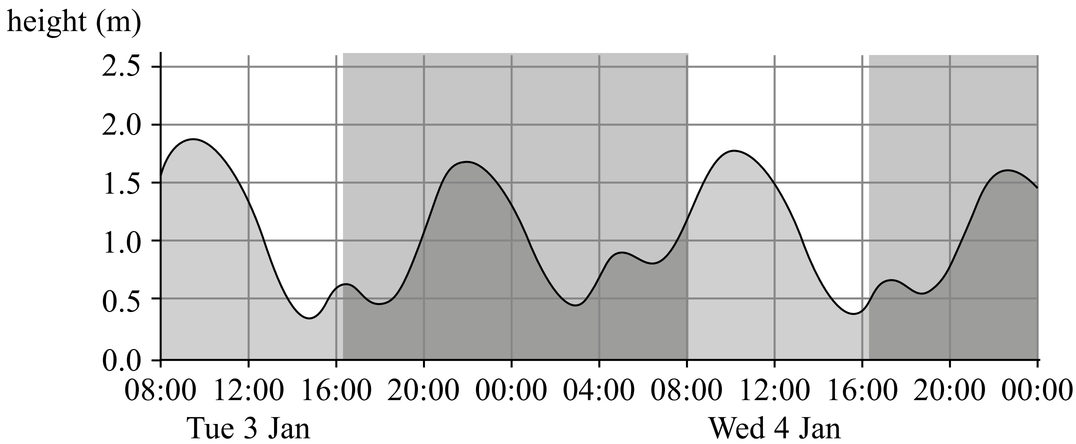
$$\mathrm{h}(x) = 45 + 15\sin x + 21\sin\left(\frac{x}{2}\right) + 25\cos\left(\frac{x}{2}\right) \qquad x \in [0, 40]$$

(a) Show that

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{(t^2 - 6t - 17)(9t^2 + 4t - 3)}{2(1 + t^2)^2}$$

where $t = \tan\left(\frac{x}{4}\right)$.

**(6)**




Source: [1]Data taken on 29th December 2016 from http://www.ukho.gov.uk/easytide/EasyTide

**Figure 2**

Figure 2 shows a graph of predicted tide heights, in metres, for Portland harbour from 08:00 on the 3rd January 2017 to the end of the 4th January 2017[1].

The graph of $k$h($x$), where $k$ is a constant and $x$ is the number of hours after 08:00 on 3rd of January, can be used to model the predicted tide heights, in metres, for this period of time.

(b) (i) Suggest a value of $k$ that could be used for the graph of $k$h($x$) to form a suitable model.

(ii) Why may such a model be suitable to predict the times when the tide heights are at their peaks, but not to predict the heights of these peaks?

**(3)**

(c) Use Figure 2 and the result of part (a) to estimate, to the nearest minute, the time of the highest tide height on the 4th January 2017.

**(6)**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **8(a)** | $\mathrm{h}(x)=45+15\sin x+21\sin\left(\frac{x}{2}\right)+25\cos\left(\frac{x}{2}\right)$ | | |
| | $\frac{\mathrm{dh}}{\mathrm{d}x}=15\cos x+\frac{21}{2}\cos\left(\frac{x}{2}\right)-\frac{25}{2}\sin\left(\frac{x}{2}\right)$ | M1 | 1.1b |
| | $\frac{\mathrm{dh}}{\mathrm{d}x}=...+..\frac{1-t^2}{1+t^2}-..\frac{2t}{1+t^2}$ | M1 | 1.1a |
| | e.g. $\frac{\mathrm{d}h}{\mathrm{d}x}=..\left(2\left(\frac{1-t^2}{1+t^2}\right)^2-1\right)+...$ **or** $\frac{\mathrm{d}h}{\mathrm{d}x}=..\frac{1-\left(\frac{2t}{1-t^2}\right)^2}{1+\left(\frac{2t}{1-t^2}\right)^2}+...$ | M1 | 3.1a |
| | e.g. $\frac{\mathrm{d}h}{\mathrm{d}x}=15\left(2\left(\frac{1-t^2}{1+t^2}\right)^2-1\right)+\frac{21}{2}\left(\frac{1-t^2}{1+t^2}\right)-\frac{25}{2}\left(\frac{2t}{1+t^2}\right)$ | A1 | 1.1b |
| | $...=\frac{15[4(1-t^2)^2-2(1+t^2)^2]+21(1-t^2)(1+t^2)-50t(1+t^2)}{2(1+t^2)^2}$ x | M1 | 2.1 |
| | $...=\frac{9t^4-50t^3-180t^2-50t+51}{2(1+t^2)^2}=\frac{(t^2-6t-17)(9t^2+4t-3)}{2(1+t^2)^2}$ * | A1* | 2.1 |
| | | **(6)** | |
| | **8(a) Alternative**<br>$\mathrm{h}(x)=...+21\left(\frac{2t}{1+t^2}\right)+25\left(\frac{1-t^2}{1+t^2}\right)$ | M1 | 1.1a |
| | $=...+15\left[2\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)\right]+...$ **or** $=...+15\left(\frac{2\left(\frac{2t}{1-t^2}\right)}{1+\left(\frac{2t}{1-t^2}\right)^2}\right)+...$ | M1 | 2.1 |
| | $\mathrm{h}(x)=45+\frac{15(4t(1-t^2))+42t(1+t^2)+25(1-t^4)}{(1+t^2)^2}$ | M1 | 1.1b |
| | $\mathrm{h}(x)=45-\frac{25t^4+18t^3-102t-25}{(1+t^2)^2}$ or $\frac{20t^4-18t^3+90t^2+102t+70}{(1+t^2)^2}$ | A1 | 1.1b |
| | $\frac{\mathrm{d}h}{\mathrm{d}x}=\frac{\mathrm{d}h}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{('u')(1+t^2)^2-('u')(4t(1+t^2))}{(1+t^2)^4}\times\frac{1}{4}(1+t^2)$ | M1 | 3.1a |
| | $...=\frac{9t^4-50t^3-180t^2-50t+51}{2(1+t^2)^2}=\frac{(t^2-6t-17)(9t^2+4t-3)}{2(1+t^2)^2}$ * | A1* | 2.1 |
| | | **(6)** | |

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| 8(b)(i) | Accept any value between $\frac{1}{40}=0.025$ and $\frac{1}{60}\approx 0.167$ inclusive | B1 | 3.3 |
| (ii) | Suitable for times since the graphs both oscillate bi-modally with about the same periodicity | B1 | 3.4 |
| | Not suitable for predicting heights since the heights of the peaks vary over time, but the graph of h($x$) has fixed peak height | B1 | 3.5b |
| | | **(3)** | |
| 8(c) | Solves at least one of the quadratics<br>$t=\frac{6\pm\sqrt{36-4\times1\times17}}{2} = 3\pm\sqrt{26}$<br>or $t=\frac{-4\pm\sqrt{16-4\times9\times(-3)}}{18} = \frac{-2\pm\sqrt{31}}{9}$ | **M1** | 1.1b |
| | Finds corresponding $x$ values, $x=4\tan^{-1}(t)$ for at least one value of t from the $9t^2+4t-3$ factor | **M1** | 1.1b |
| | One correct value for these x<br>e.g. $x=$ awrt $-2.797$ or $9.770, 1.510$ | **A1** | 1.1b |
| | Maximum peak height occurs at smallest positive value of x, from first graph, but the third of these peaks needed,<br>So $t = 1.509\ldots + 8\pi = 26.642$ is the is the required time | **M1** | 3.4 |
| | $x = 26.642$ corresponds to 26 hours and 39 minutes (nearest minute) after 08:00 on 3rd January<br>(Allow if a different greatest peak height used) | **M1** | 3.4 |
| | Time of greatest tide height is approximately 10:39 (am)<br>(also allow 10:38 or 10:40) | **A1** | 3.2a |
| | | **(6)** | |
| | | | **(15 marks)** |

**Notes:**

**(a)**

**M1:** Differentiates h($x$)

**M1:** Applies $t$-substitution to both $\left(\frac{x}{2}\right)$ terms with their coefficients

**M1:** Forms a correct expression in $t$ for the cos $x$ term, using double angle formula and $t$-substitution, or double '$t$'-substitution

**A1:** Fully correct expression in $t$ for $\frac{\mathrm{d}h}{\mathrm{d}x}$

**M1:** Gets all terms over the correct common factor. Numerators must be appropriate for their terms

**A1*:** Achieves the correct answer via expression with correct quartic numerator before factorisation

| Question 8 notes continued: |
|---|
| **Alternative:**<br>**(a)**<br>**M1:** Applies $t$-substitution to both $\left(\frac{x}{2}\right)$ terms<br>**M1:** Forms a correct expression in $t$ for the sin $x$ term, using double angle formula and $t$-substitution, or double '$t$'-substitution<br>**M1:** Gets all terms in $t$ over the correct common factor. Numerators must be appropriate for their terms. May include the constant term too<br>**A1:** Fully correct expression in $t$ for $\mathrm{h}(x)$<br>**M1:** Differentiates, using both chain rule and quotient rule with their '$u$'<br>**A1*:** Achieves the correct answer via expression with correct quartic numerator before factorisation<br>**Note:** The individual terms may be differentiated before putting over a common denominator. In this case score the third M for differentiating with chain rule and quotient rule, then r return to the original scheme |
| **(b)(i)**<br>**B1:** Any value between $\frac{1}{40}$ (e.g. taking $\mathrm{h}(0)$ as reference point) **or** $\frac{1}{60}$ (taking lower peaks as reference)<br>**NB:** **Taking high peak as reference gives $\frac{1}{50}$** |
| **(b)(ii)**<br>**B1:** Should mention both the bimodal nature and periodicity for the actual data match the graph of h<br>**B1:** Mentions that the heights of peaks vary in each oscillation |
| **(c)**<br>**M1:** Solves (at least) one of the quadratic equations in the numerator<br>**M1:** Must be attempting to solve the quadratic factor from which the solution comes $9t^2 + 4t - 3$ and using $t = \tan\left(\frac{x}{4}\right)$ to find a corresponding value for $x$<br>**A1:** At least one correct $x$ value from solving the requisite quadratic: awrt any of – 2.797, 1.510, 9.770, 14.076, 22.336, 26.642, 34.902 or 39.208<br>**M1:** Uses graph of h to pick out their $x$ = 26.642 as the time corresponding to the third of the higher peaks, which is the highest of the peaks on 4th January on the tide height graph. As per scheme or allow if all times listed and correct one picked<br>**M1:** Finds the time for one of the values of $t$ corresponding to the highest peaks. E.g. 1.5096...~ 09:31 (3rd January) or 14.076... ~ 22:05 (3rd January) or 26.642…~ 10:39 (4th January) or 39.208…~ 23:13 (4th January). (Only follow through on use of the smallest positive $t$ solution $+ 4k\pi$ )<br>**A1:** Time of greatest tide height on 4th January is approximately 10:39. Also allow 10:38 or 10:40 |

**3.** (a) Using the substitution $t = \tan\left(\frac{x}{2}\right)$ show that the equation

$$4\tan x + 3\cot\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right) = 0$$

can be written as

$$3t^4 - 8t^2 - 3 = 0$$

**(3)**

(b) Hence solve, for $-2\pi < x \leqslant 2\pi$

$$4\tan x + 3\cot\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right) = 0$$

giving your answers in terms of $\pi$

**(4)**

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| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **3(a)** | $4\tan x + 3\cot\left(\frac{x}{2}\right)\ \sec^2\left(\frac{x}{2}\right)$ <br> $= 4\tan x + \frac{3}{\tan\left(\frac{x}{2}\right)}\left(1 + \tan^2\left(\frac{x}{2}\right)\right)$ <br> $= 4\left(\frac{2t}{1-t^2}\right) + \frac{3}{t}(1 + t^2)$ | M1 | 2.1 |
| | $\left(\frac{8t}{1-t^2}\right) + \frac{3(1+t^2)}{t} = 0$ <br> $8t^2 + 3(1+t^2)(1-t^2) = 0$ or $\frac{8t^2+3(1+t^2)(1-t^2)}{t(1-t^2)} = 0$ | M1 | 1.1b |
| | $3t^4 - 8t^2 - 3 = 0$ * | A1* | 1.1b |
| | | **(3)** | |
| **(b)** | Solves quadratic for $t^2$ by factorising, quadratic formula, calculator <br> $(3t^2+1)(t^2-3) = 0$ , $t^2 = \frac{8 \pm \sqrt{(-8)^2-4(3)(-3)}}{2(3)}$ <br> $t^2 = 3,\ t^2 = -\frac{1}{3}$ <br> leading to value for $t$ = …. | M1 | 3.1a |
| | $t = \pm\sqrt{3}$ | A1 | 1.1b |
| | Finds two correct values for $x = -\frac{4\pi}{3}\ ,\ -\frac{2\pi}{3}\ ,\ \frac{2\pi}{3}\ ,\ \frac{4\pi}{3}$ | M1 | 1.1b |
| | All correct values for $x = -\frac{4\pi}{3}\ ,\ -\frac{2\pi}{3}\ ,\ \frac{2\pi}{3}\ ,\ \frac{4\pi}{3}$ | A1 | 2.2a |
| | | **(4)** | |
| | | | **(7 marks)** |

**Notes:**

**(a)**

**M1:** Expresses $3\cot\left(\frac{x}{2}\right)\ \sec^2\left(\frac{x}{2}\right)$ in terms of $\tan\left(\frac{x}{2}\right)$ and uses $t$ substitutions to obtain an expression in terms of $t$ only

**M1:** Multiplies through by $t$ and $(1 - t^2)$ or forms a common denominator

**A1*:** $3t^4 - 8t^2 - 3 = 0$ cso

**(c)**

**M1:** Solve the quadratic in $t^2$ leading to a value for $t$

**A1:** Correct values for $t = \pm\sqrt{3}$

**M1:** Finds two correct values for $x = -\frac{4\pi}{3}\ ,\ -\frac{2\pi}{3}\ ,\ \frac{2\pi}{3}\ ,\ \frac{4\pi}{3}$

**A1:** All correct values for $x = -\frac{4\pi}{3}\ ,\ -\frac{2\pi}{3}\ ,\ \frac{2\pi}{3}\ ,\ \frac{4\pi}{3}$

**2.** During 2029, the number of hours of daylight per day in London, $H$, is modelled by the equation

$$H = 0.3\sin\left(\frac{x}{60}\right) - 4\cos\left(\frac{x}{60}\right) + 11.5 \qquad 0 \leqslant x < 365$$

where $x$ is the number of days after 1st January 2029 and the angle is in radians.

(a) Show that, according to the model, the number of hours of daylight in London on the 31st January 2029 will be 8.13 to 3 significant figures.

**(1)**

(b) Use the substitution $t = \tan\left(\frac{x}{120}\right)$ to show that $H$ can be written as

$$H = \frac{at^2 + bt + c}{1 + t^2}$$

where $a$, $b$ and $c$ are constants to be determined.

**(2)**

(c) Hence determine, according to the model, the date of the first day of 2029 when there will be at least 12 hours of daylight in London.

**(4)**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **2(a)** | $(H=)0.3\sin\left(\frac{30}{60}\right)-4\cos\left(\frac{30}{60}\right)+11.5=8.13$ {hours}* | B1* | 3.4 |
| | | **(1)** | |
| **(b)** | Substitutes $\sin\left(\frac{x}{60}\right)=\frac{2t}{1+t^2}$ and $\cos\left(\frac{x}{60}\right)=\frac{1-t^2}{1+t^2}$ into $H$ $$(H=)0.3\left(\frac{2t}{1+t^2}\right)-4\left(\frac{1-t^2}{1+t^2}\right)+11.5$$ | M1 | 1.1b |
| | $$(H=)\frac{0.6t-4+4t^2+11.5(1+t^2)}{1+t^2}=\frac{15.5t^2+0.6t+7.5}{1+t^2}$$ | A1 | 2.1 |
| | | **(2)** | |
| **(c)** | $$H=\frac{15.5t^2+0.6t+7.5}{1+t^2}=12\Rightarrow 3.5t^2+0.6t-4.5=0$$ | M1 | 3.4 |
| | $\Rightarrow t=\frac{-0.6\pm\sqrt{0.6^2-4(3.5)(-4.5)}}{7}$ $=\ldots(1.051\ldots,-1.222\ldots)\Rightarrow x=120\tan^{-1}("1.051..")$ $=\ldots(97.254..)$ | dM1 | 3.1b |
| | $x=$ awrt 97 | A1 | 1.1b |
| | 8th or 9th April | A1 | 3.2a |
| | | **(4)** | |
| | | **(7 marks)** | |

**Notes:**

**(a)**

**B1*:** Uses $x$ = 30 to show that $H$ = 8.13. Accept $x$ = 30 seen substituted followed by 8.13, or $x$ =30 identified followed by 8.133… before rounding to 8.13.

**(b)**

**M1:** Uses the correct $t$-formulae $\sin\left(\frac{x}{60}\right)=\frac{2t}{1+t^2}$ and $\cos\left(\frac{x}{60}\right)=\frac{1-t^2}{1+t^2}$, attempts to substitute into $H$.

**A1:** Fully correct method, expresses as a single fraction with a denominator of $1+t^2$ to achieve $H=\frac{15.5t^2+0.6t+7.5}{1+t^2}$ (oe with fractions or accept values for $a$, $b$ and c stated).

**(c)**

**M1:** Sets $H$ = 12 (or any inequality in between) and rearranges to form a a quadratic equation for $t$.

**dM1:** Dependent on the previous method mark. Solves the quadratic by any means (accept one correct answer for their quadratic if no method shown) and uses this to find a value for $x$.

**A1:** Correct value for $x=$ awrt 97 or accept 98 following a correct value for $t$.

**A1:** Correct day of the year. Accept 8th or 9th April following awrt 97 from a correct method.

**Note:** Question says hence, so answers by graphical methods or trial and improvement are not acceptable for full credit. They can score a SC M0dM0A0B1 for achieving a correct date.

**Answer ALL questions. Write your answers in the spaces provided.**

**1.** (a) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that the equation

$$5\sin x + 12\cos x = 2$$

can be written in the form

$$7t^2 - 5t - 5 = 0$$

**(3)**

(b) Hence solve, for $-180° < x < 180°$, the equation

$$5\sin x + 12\cos x = 2$$

giving your answers to one decimal place.

**(4)**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **1** | $t=\tan\left(\frac{x}{2}\right),\quad 5\sin x+12\cos x=2 \Rightarrow 7t^2-5t-5=0$ | | |
| **(a)** | $\{5\sin x+12\cos x=\}\ 5\left(\frac{2t}{1+t^2}\right)+12\left(\frac{1-t^2}{1+t^2}\right)$ | M1 | 1.1b |
| | $5\left(\frac{2t}{1+t^2}\right)+12\left(\frac{1-t^2}{1+t^2}\right)=2 \Rightarrow 5(2t)+12(1-t^2)=2(1+t^2)$ | M1 | 1.1b |
| | $7t^2-5t-5=0$ * | A1* | 2.1 |
| | | **(3)** | |
| **(b)** | $t=\frac{5\pm\sqrt{5^2-4(7)(-5)}}{2(7)}\quad \left\{=\frac{5\pm\sqrt{165}}{14}=1.2746...,\ -0.5603...\right\}$ | M1 | 1.1a |
| | $\frac{x}{2}=\arctan\left(\frac{5+\sqrt{165}}{14}\right)$ or $\frac{x}{2}=\arctan\left(\frac{5-\sqrt{165}}{14}\right)$ **and** $\Rightarrow x=...$ | M1 | 3.1a |
| | $x=$ awrt $104^{\circ}$ **or** $x=$ any answer in the range $[-58.6^{\circ},\ -58^{\circ}]$ | A1 | 1.1b |
| | $x=103.8^{\circ}$ **and** $x=-58.5^{\circ}$ | A1 | 1.1b |
| | | **(4)** | |
| | | | **(7 marks)** |

**Notes**

| | |
|---|---|
| **(a)** | |
| **M1:** | Uses at least one of $\sin x=\frac{2t}{1+t^2}$ or $\cos x=\frac{1-t^2}{1+t^2}$ to express $5\sin x+12\cos x$ in terms of $t$ only |
| **M1:** | Uses both correct formula $\sin x=\frac{2t}{1+t^2}$ and $\cos x=\frac{1-t^2}{1+t^2}$ in $5\sin x+12\cos x$, equates their expression to 2 and eliminates the fractions |
| **A1*:** | Collects terms to one side and simplifies to obtain the printed answer |
| **(b)** | |
| **M1:** | Selects a correct process (e.g. using the quadratic formula, completing the square or calculator approach) to solve $7t^2-5t-5=0$ |
| **Note:** | Allow 1st M1 for at least one of awrt 1.3 or awrt $-0.6$ or for a correct exact value of $t$ |
| **Note:** | Do not allow an attempt at factorisation of $7t^2-5t-5$ for the 1st M1 |
| **M1:** | Adopts a correct ***applied*** strategy of taking arctan(their found $t$) and multiplying the result by 2 to obtain at least one value for $x$ within the range $-180^{\circ}<x<180^{\circ}$ (or in radians $-\pi<x<\pi$ ) |
| **A1:** | See scheme |
| **A1:** | For both 103.8 and $-58.5$ |
| **Note:** | Give final A0 for extra solutions given within the range $-180^{\circ}<x<180^{\circ}$ |
| **Note:** | Ignore extra solutions outside the range $-180^{\circ}<x<180^{\circ}$ for the final A mark |
| **Note:** | In degrees, $\frac{x}{2}=\{51.88...,\ -128.11...,\ -29.26...,\ 150.73...\}$ |
| **Note:** | Working in radians gives $\frac{x}{2}=\{0.905...,\ -0.510...\}\Rightarrow x=\{1.81...,\ -1.02...\}$ |
| **Note:** | Give 2nd M0 for $\frac{x}{2}=\{51.88...,\ -29.26...\}\Rightarrow x=\{25.9,\ -14.6\}$ |

1. (a) Write down the $t$-formula for $\sin x$.

**(1)**

(b) Use the answer to part (a)

(i) to find the exact value of $\sin x$ when

$$\tan\left(\frac{x}{2}\right) = \sqrt{2}$$

(ii) to show that

$$\cos x = \frac{1 - t^2}{1 + t^2}$$

**(4)**

(c) Use the $t$-formulae to solve for $0 < \theta \leqslant 360°$

$$7\sin\theta + 9\cos\theta + 3 = 0$$

giving your answers to one decimal place.

**(4)**

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| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **1 (a)** | $\{\sin x =\}\ \dfrac{2t}{1+t^2}$ | B1 | 1.2 |
| | | **(1)** | |
| **(b)(i)** | $\left\{\tan\left(\dfrac{x}{2}\right)=\sqrt{2}\ \Rightarrow\ t=\sqrt{2}\Rightarrow\right\}\ \sin x = \dfrac{2(\sqrt{2})}{1+(\sqrt{2})^2}$ or $\dfrac{2(\sqrt{2})}{1+2}$ | M1 | 1.1b |
| | $\sin x = \dfrac{2}{3}\sqrt{2}$ or $\dfrac{1}{3}\sqrt{8}$ or $\sqrt{\dfrac{8}{9}}$ | A1 | 1.1b |
| | | **(2)** | |
| **(ii) Way 1** | $\left\{\cos x \equiv \dfrac{\sin x}{\tan x}\ \Rightarrow\right\}\ \cos x = \dfrac{\frac{2t}{1+t^2}}{\frac{2t}{1-t^2}}\ ;\ = \dfrac{1-t^2}{1+t^2}$ * cso | M1; | 1.1b |
| | | A1* | 2.1 |
| | | **(2)** | |
| **(ii) Way 2** | $\left\{\tan x \equiv \dfrac{\sin x}{\cos x}\ \Rightarrow\right\}\ \dfrac{2t}{1-t^2} = \dfrac{\frac{2t}{1+t^2}}{\cos x}\ ;\ \Rightarrow\ \cos x = \dfrac{1-t^2}{1+t^2}$ * cso | M1; | 1.1b |
| | | A1* | 2.1 |
| | | **(2)** | |
| **(ii) Way 3** | $\{\sin^2 x+\cos^2 x \equiv 1 \Rightarrow\}\ \left(\dfrac{2t}{1+t^2}\right)^2 + \cos^2 x = 1$ | M1 | 1.1b |
| | $\cos^2 x = 1-\left(\dfrac{2t}{1+t^2}\right)^2 = \dfrac{(1+t^2)^2-4t^2}{(1+t^2)^2} = \dfrac{1-2t^2+t^4}{(1+t^2)^2} = \dfrac{(1-t^2)^2}{(1+t^2)^2}$ <br> $\Rightarrow \cos x = \dfrac{1-t^2}{1+t^2}$ * cso | A1 | 2.1 |
| | | **(2)** | |
| **(ii) Way 4** | $\{o^2+a^2=h^2\ \Rightarrow\}\ (2t)^2+a^2=(1+t^2)^2$ | M1 | 1.1b |
| | $a^2=(1+t^2)^2-(2t)^2 = 1-2t^2+t^4 = (1-t^2)^2$ <br> $a=1-t^2\ \Rightarrow\ \cos x = \dfrac{1-t^2}{1+t^2}$ * cso | A1 | 2.1 |
| | | **(2)** | |
| **(c)** | $\{7\sin\theta+9\cos\theta+3=\}\ \ 7\left(\dfrac{2t}{1+t^2}\right)+9\left(\dfrac{1-t^2}{1+t^2}\right)+3$ | M1 | 1.1b |
| | $7\left(\dfrac{2t}{1+t^2}\right)+9\left(\dfrac{1-t^2}{1+t^2}\right)+3=0\ \Rightarrow\ 14t+9-9t^2+3+3t^2=0$ <br> $\Rightarrow 6t^2-14t-12=0 \Rightarrow 3t^2-7t-6=0\ \Rightarrow (t-3)(3t+2)=0 \Rightarrow t=\dots$ | M1 | 1.1b |
| | **Either** $\left\{t=3\ \Rightarrow\ \dfrac{\theta}{2}=\arctan(3)\ \Rightarrow\right\}\ \theta = 2\arctan(3)$ **or** <br> $\left\{t=-\dfrac{2}{3}\ \Rightarrow\ \dfrac{\theta}{2}=180°+\arctan\left(-\dfrac{2}{3}\right)\ \Rightarrow\right\}\ \theta = 2\left(180°+\arctan\left(-\dfrac{2}{3}\right)\right)$ | M1 | 1.1b |
| | $\dfrac{\theta}{2}=\{71.5650\ldots,\ 146.3099\ldots\}\ \Rightarrow\ \theta = \{143.1301\ldots,\ 292.6198\ldots\}$ | | |
| | $\theta = 143.1°,\ 292.6°$ (1dp) | A1 | 1.1b |
| | | **(4)** | |
| | | | **(9 marks)** |

| Notes for Question 1 | |
|---|---|
| **(a)** | |
| **B1:** | See scheme |
| **(b)(i)** | |
| **M1:** | Complete substitution of $t=\sqrt{2}$ into their expression from part (a) |
| **A1:** | Correct exact answer. See scheme. |
| **Note:** | Give M0 A0 for writing down the correct exact answer without any evidence of substituting $t=\sqrt{2}$ into $\sin x = \frac{2t}{1+t^2}$ |
| **Note:** | For reference, $\sin x = \frac{2}{3}\sqrt{2} = 0.9428...$ |
| **(b)(ii)** | **Way 1, Way 2 and Way 3** |
| **M1:** | Uses a correct trigonometric identity (or correct trigonometric identities) to find a correct expression which connects only $\cos x$ (or $\cos^2 x$) and $t$ |
| **A1*:** | Correct proof |
| **(b)(ii)** | **Way 4** |
| **M1:** | Uses $\sin x = \frac{o}{h}$ and a correct Pythagoras method to express the adjacent edge of a triangle in terms of *t*. |
| **A1*** | Correct proof |
| **(c)** | |
| **M1:** | Uses at least one of $\sin\theta = \frac{2t}{1+t^2}$ or $\cos\theta = \frac{1-t^2}{1+t^2}$ to express $7\sin\theta + 9\cos\theta + 3$ in terms of *t* only |
| **M1:** | Uses both correct formula $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$ in $7\sin\theta + 9\cos\theta + 3 = 0$, multiplies both sides by $1+t^2$, forms a 3TQ and uses a correct method (e.g. using the quadratic formula, completing the square or a calculator approach) for solving their 3TQ to give $t = ...$ |
| **M1:** | Uses both correct formula $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$ in $7\sin\theta + 9\cos\theta + 3 = 0$, adopts a correct *applied* strategy to find at least one value of $\theta$ within the range $0 < \theta \le 360°$ (or in radians $0 < \theta \le 2\pi$) such that either<br>• $\theta = 2\arctan(\text{their found } t)$, where their found $t > 0$<br>• $\theta = 2(180° + \arctan(\text{their found } t))$, where their found $t < 0$<br>• $\theta = 2(180° - \arctan\lvert\text{their found } t\rvert)$, where their found $t < 0$ |
| **A1:** | Correct answer only of $\theta = 143.1°, 292.6°$ |
| **Note:** | Give A0 for extra solutions given within the range $0 < \theta \le 360°$ |
| **Note:** | Ignore extra solutions outside the range $0 < \theta \le 360°$ for the A mark |
| **Note:** | Give 3rd M0 for $\frac{\theta}{2} = \{71.565..., 146.309...\}$ without attempting to find $\theta$ |
| **Note** | Give 3rd M0 for $\frac{\theta}{2} = \{71.565..., 146.309...\} \Rightarrow \theta = \{35.782..., 73.154...\}$ |
| **Note:** | In degrees, $\frac{\theta}{2} = \{71.565...,\ 251.565...,\ -33.690...,\ 146.309...\}$ |
| **Note:** | Working in radians gives $\frac{\theta}{2} = \{1.249..., 2.553...\} \Rightarrow \theta = \{2.498..., 5.107...\}$ |

**3.** (i) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to prove that

$$\cot x + \tan\left(\frac{x}{2}\right) = \operatorname{cosec} x \qquad x \neq n\pi,\ n \in \mathbb{Z}$$

**(2)**

(ii)

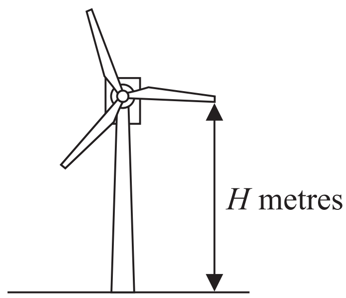


**Figure 1**

An engineer models the vertical height above the ground of the tip of one blade of a wind turbine, shown in Figure 1. The ground is assumed to be horizontal.

The vertical height of the tip of the blade above the ground, $H$ metres, at time $x$ seconds after the wind turbine has reached its constant operating speed, is modelled by the equation

$$H = 90 - 30\cos(120x)^\circ - 40\sin(120x)^\circ \qquad \text{(I)}$$

(a) Show that $H = 60$ when $x = 0$

**(1)**

Using the substitution $t = \tan(60x)^\circ$

(b) show that equation (I) can be rewritten as

$$H = \frac{120t^2 - 80t + 60}{1 + t^2}$$

**(3)**

(c) Hence find, according to the model, the value of $x$ when the tip of the blade is 100 m above the ground for the first time after the wind turbine has reached its constant operating speed.

**(5)**

| Question | Scheme | Marks | AOs |
| --- | --- | --- | --- |
| **3(i)** | $\text{lhs} = \cot x + \tan\left(\frac{x}{2}\right) = \frac{1-t^2}{2t} + t$ | M1 | 1.1a |
| | $\frac{1-t^2}{2t} + t = \frac{1+t^2}{2t}\left(= \frac{1}{\sin x}\right) = \text{cosec } x*$ | A1* | 2.1 |
| | | **(2)** | |
| **(ii)(a)** | $x = 0 \Rightarrow H = 90 - 30\cos(0) - 40\sin(0) = 90 - 30 = 60$ | B1 | 1.1b |
| | | **(1)** | |
| **(b)** | $H = 90 - 30\cos 120x - 40\sin 120x = 90 - 30\left(\frac{1-t^2}{1+t^2}\right) - 40\left(\frac{2t}{1+t^2}\right)$ | M1 | 1.1b |
| | $= \frac{90 + 90t^2 - 30 + 30t^2 - 80t}{1+t^2}$ | M1 | 1.1b |
| | $= \frac{120t^2 - 80t + 60}{1+t^2}*$ | A1* | 2.1 |
| | | **(3)** | |
| **(c)** | $\frac{120t^2 - 80t + 60}{1+t^2} = 100 \Rightarrow 120t^2 - 80t + 60 = 100 + 100t^2$ | M1 | 3.4 |
| | $20t^2 - 80t - 40 = 0$ | A1 | 1.1b |
| | $t = \frac{4 \pm \sqrt{16+8}}{2} \Rightarrow 60x = \tan^{-1}\left(2+\sqrt{6}\right)$ or $60x = \tan^{-1}\left(2-\sqrt{6}\right)$ | M1 | 3.4 |
| | $60x = \tan^{-1}\left(2+\sqrt{6}\right) = 77.33... \Rightarrow x = ...$ | dM1 | 3.1b |
| | $x = 1.29$ | A1 | 3.2a |
| | | **(5)** | |
| | | | **(11 marks)** |

**Notes**

(ii)
M1: Selects the correct expression for cot $x$ in terms of $t$ and substitutes this and $t$ into the lhs
A1*: Fully correct proof. Allow correct work leading to $\frac{1+t^2}{2t} = \text{cosec } x$
(ii)(a)
B1: Demonstrates that when $x = 0$, $H = 60$
(b)
M1: Uses the correct formulae to obtain $H$ in terms of $t$
M1: Correct method to obtain a common denominator
A1*: Collects terms and simplifies to obtain the printed answer with no errors
(c)
M1: Uses $H = 100$ with the model and multiplies up to obtain a quadratic equation in $t$
A1: Correct 3TQ
M1: Solves their 3TQ in $t$ and proceeds to obtain values of $60x$ as suggested by the model
M1: A fully correct strategy to identify the required value of $x$ from the positive root of the quadratic equation in $t$
A1: awrt 1.29
Attempts in radians can score all but the final mark in (c). (Gives $60x = 1.3…$ etc.)

**3.** On a particular day, the depth of water in a river estuary at a specific location is modelled by the equation

$$D = 2\sin\left(\frac{x}{3}\right) + 3\cos\left(\frac{x}{3}\right) + 6 \qquad 0 \leqslant x \leqslant 7\pi \qquad \text{(I)}$$

where the depth of water is $D$ metres at time $x$ hours after midnight on that day.

(a) Write down the depth of water at midnight, according to the model.

**(1)**

Using the substitution $t = \tan\left(\frac{x}{6}\right)$

(b) show that equation (I) can be re-written as

$$D = \frac{3t^2 + 4t + 9}{1 + t^2}$$

**(3)**

(c) Hence determine, according to the model, the time after midnight when the depth of water is 5 metres for the first time. Give your answer to the nearest minute.

**(5)**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **3(a)** | $x=0 \Rightarrow D=2\sin(0)+3\cos(0)+6=6+3=9\text{ m}$ | B1 | 3.4 |
| | | **(1)** | |
| **(b)** | $D=2\left(\dfrac{2t}{1+t^2}\right)+3\left(\dfrac{1-t^2}{1+t^2}\right)+6$ | M1 | 1.1b |
| | $=\dfrac{4t+3-3t^2+6+6t^2}{1+t^2}$ | M1 | 1.1b |
| | $=\dfrac{3t^2+4t+9}{1+t^2}$ * | A1* | 2.1 |
| | | **(3)** | |
| **(c)** | $\dfrac{3t^2+4t+9}{1+t^2}=5 \Rightarrow 3t^2+4t+9=5+5t^2$ | M1 | 3.4 |
| | $t^2-2t-2=0$ | A1 | 1.1b |
| | $t=\dfrac{2\pm\sqrt{4+8}}{2} \Rightarrow \dfrac{x}{6}=\tan^{-1}\left(1+\sqrt{3}\right)$ or $\dfrac{x}{6}=\tan^{-1}\left(1-\sqrt{3}\right)$ | M1 | 3.4 |
| | $\dfrac{x}{6}=\tan^{-1}\left(1+\sqrt{3}\right)=1.21... \Rightarrow x=...$ | M1 | 3.1b |
| | 0719 or 07:19 am | A1 | 3.2a |
| | | **(5)** | |
| | | | **(9 marks)** |

**Notes**

(a)
B1: Obtains the correct depth of 9 m (must include units)
(b)
M1: Uses the correct formulae to obtain *D* in terms of *t*
M1: Correct method to obtain a common denominator
A1*: Collects terms and simplifies to obtain the printed answer with no errors
(c)
M1: Uses *D* = 5 with the model and multiplies up to obtain a quadratic equation in *t*
A1: Correct 3TQ
M1: Solves their 3TQ in *t* and proceeds to obtain values of $\dfrac{x}{6}$ as suggested by the model
M1: A fully correct strategy to identify the required value of *x* from the positive root of the quadratic equation in *t*
A1: Correct time. Allow e.g. 439 minutes after midnight.

**3.** (a) Use $t = \tan\dfrac{\theta}{2}$ to show that, where both sides are defined

$$\frac{29 - 21\sec\theta}{20 - 21\tan\theta} \equiv \frac{5t + 2}{2t + 5}$$

**(4)**

(b) Hence, again using $t = \tan\dfrac{\theta}{2}$, prove that, where both sides are defined

$$\frac{20 + 21\tan\theta}{29 + 21\sec\theta} \equiv \frac{29 - 21\sec\theta}{20 - 21\tan\theta}$$

**(3)**

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| Question | Scheme | Marks | AOs |
| --- | --- | --- | --- |
| 3(a) | Sight of $\sec\theta=\dfrac{1+t^2}{1-t^2}$ and $\tan\theta=\dfrac{2t}{1-t^2}$ at least once each. | B1 | 1.2 |
| | $\dfrac{29-21\sec\theta}{20-21\tan\theta}=\dfrac{29-21\left(\dfrac{1+t^2}{1-t^2}\right)}{20-21\left(\dfrac{2t}{1-t^2}\right)}$ | M1 | 1.1b |
| | $=\dfrac{29\left(1-t^2\right)-21\left(1+t^2\right)}{20\left(1-t^2\right)-21(2t)}=\dfrac{8-50t^2}{-20t^2-42t+20}$ <br> or <br> $=\dfrac{\dfrac{29\left(1-t^2\right)-21\left(1+t^2\right)}{\left(1-t^2\right)}}{\dfrac{20\left(1-t^2\right)-21(2t)}{\left(1-t^2\right)}}=\dfrac{8-50t^2}{-20t^2-42t+20}$ | M1 | 2.1 |
| | $=\dfrac{-2(5t-2)(5t+2)}{-2(5t-2)(2t+5)}=\dfrac{5t+2}{2t+5}$ * cso <br> $=\dfrac{2(2+5t)(2-5t)}{2(2-5t)(5+2t)}=\dfrac{5t+2}{2t+5}$ * cso | A1* | 1.1b |
| | | (4) | |
| (b) | $\dfrac{20+21\tan\theta}{29+21\sec\theta}=\dfrac{20+21\left(\dfrac{2t}{1-t^2}\right)}{29+21\left(\dfrac{1+t^2}{1-t^2}\right)}$ <br> $=\dfrac{20\left(1-t^2\right)+21(2t)}{29\left(1-t^2\right)+21\left(1+t^2\right)}$ | M1 | 2.1 |
| | $\dfrac{20+42t-20t^2}{50-8t^2}=\dfrac{2(5-2t)(5t+2)}{2(5-2t)(5+2t)}=\ldots$ <br> $\dfrac{20+42t-20t^2}{50-8t^2}=\dfrac{-2(2t-5)(5t+2)}{-2(2t+5)(2t-5)}=\ldots$ | M1 | 1.1b |
| | Achieves form correct working $=\dfrac{5t+2}{2t+5}$ * <br> Then concludes <br> hence the result is true <br> or <br> = RHS <br> or <br> $=\dfrac{5t+2}{2t+5}=\dfrac{29-21\sec\theta}{20-21\tan\theta}$ <br> or | A1* | 2.2a |

| | | | |
|---|---|---|---|
| | $\dfrac{20+21\tan\theta}{29+21\sec\theta}=\dfrac{29-21\sec\theta}{20-21\tan\theta}$<br>LHS = RHS | | |
| | | **(3)** | |

**(7 marks)**

**Notes:**

**(a)**

**B1:** Uses the correct identities at least once each. May be seen in (a) or (b)

**M1:** Substitutes their identities into the equation, they need not be correct

**M1:** Multiplies numerator and denominator by $1-t^2$ and simplifies to a quadratic with all terms collected

Special case: If they make an error with the identities award M1 if they use the correct method to write the numerator and denominator as a single fraction and then divides to achieve an expression of the form $\dfrac{a}{b}$

**A1*:** Cancels factors to achieve the given expression, with no errors seen

Condone $\dfrac{50t^2-8}{20t^2+42t-20}=\dfrac{(5t+2)(5t-2)}{(5t-2)(2t+5)}=\dfrac{5t+2}{2t+5}$ and $\dfrac{-8+50t^2}{-20t^2-42t+20}=\dfrac{(5t+2)(5t-2)}{(5t-2)(2t+5)}=\dfrac{5t+2}{2t+5}$

**(b)**

**M1:** Substitutes into LHS of equation and multiplies numerator and denominator by $1-t^2$ to work towards the other side.

**M1:** Simplifies to a quadratic with all terms collected and factorises then cancels terms in the LHS in an attempt to try and match expressions.

**A1*:** Achieves correct expressions for both sides and gives a conclusion deducing that the result is true.

M1A0 for $\dfrac{20t^2-42t-20}{8t^2-50}=\dfrac{(5t+2)(2t-5)}{(2t-5)(2t+5)}=\dfrac{5t+2}{2t+5}$ and

$\dfrac{-20t^2-42t+20}{-8+50t^2}=\dfrac{(5t+2)(2t-5)}{(2t-5)(2t+5)}=\dfrac{5t+2}{2t+5}$