Cp2Ch8 XMQs and MS

(Total: 163 marks)

1. CP1_Sample Q5 . 10 marks - CP2ch8 Modelling with differential equations
2. CP1_Sample Q9 . 12 marks - CP2ch8 Modelling with differential equations
3. CP2_Sample Q7 . 17 marks - CP2ch8 Modelling with differential equations
4. CP1_Specimen Q8 . 14 marks - CP2ch8 Modelling with differential equations
5. CP2_Specimen Q6 . 13 marks - CP2ch7 Methods in differential equations
6. CP1_2019 Q5 . 13 marks - CP2ch8 Modelling with differential equations
7. CP1_2019 Q8 . 18 marks - CP2ch8 Modelling with differential equations
8. CP1_2020 Q5 . 17 marks - CP2ch8 Modelling with differential equations
9. CP2_2020 Q3 . 14 marks - CP2ch7 Methods in differential equations
10. CP1_2021 Q6 . 12 marks - CP2ch8 Modelling with differential equations
11. CP1_2021 Q8 . 9 marks - CP2ch8 Modelling with differential equations
12. CP1_2022 Q10. 14 marks - CP2ch7 Methods in differential equations

5. A pond initially contains 1000 litres of unpolluted water.

The pond is leaking at a constant rate of 20 litres per day.

It is suspected that contaminated water flows into the pond at a constant rate of 25 litres per day and that the contaminated water contains 2 grams of pollutant in every litre of water.

It is assumed that the pollutant instantly dissolves throughout the pond upon entry.

Given that there are x grams of the pollutant in the pond after t days,

(a) show that the situation can be modelled by the differential equation,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 50 - \frac{4x}{200 + t}$$

(4)

(b) Hence find the number of grams of pollutant in the pond after 8 days.

(5)

(c) Explain how the model could be refined.

(1)

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Question	Scheme	Marks	AOs
5(a)	Pond contains 1000 + 5t litres after t days	M1	3.3
	If x is the amount of pollutant in the pond after t days Rate of pollutant out = $20\times\dfrac{x}{1000+5t}$ g per day	M1	3.3
	Rate of pollutant in = 25 × 2 g = 50g per day	B1	2.2a
	$\dfrac{dx}{dt}=50-\dfrac{4x}{200+t}$ *	A1*	1.1b
		(4)	
(b)	$I=e^{\int\frac{4}{200+t}dt}=(200+t)^4\Rightarrow x(200+t)^4=\int 50(200+t)^4\,dt$	M1	3.1b
	$x(200+t)^4=10(200+t)^5+c$	A1	1.1b
	$x=0,\ t=0\Rightarrow\ c=-3.2\times10^{12}$	M1	3.4
	$t=8\Rightarrow\ x=\ 10(200+8)-\dfrac{3.2\times10^{12}}{(200+8)^4}$	M1	1.1b
	= 370g	A1	2.2b
		(5)	
(c)	e.g. • The model should take into account the fact that the pollutant does not dissolve throughout the pond upon entry • The rate of leaking could be made to vary with the volume of water in the pond	B1	3.5c
		(1)	
		(10 marks)	

Notes:

(a)

M1: Forms an expression of the form 1000 + kt for the volume of water in the pond at time t

M1: Expresses the amount of pollutant out in terms of x and t

B1: Correct interpretation for pollutant entering the pond

A1*: Puts all the components together to form the correct differential equation

(b)

M1: Uses the model to find the integrating factor and attempts solution of their differential equation

A1: Correct solution

M1: Interprets the initial conditions to find the constant of integration

M1: Uses their solution to the problem to find the amount of pollutant after 8 days

A1: Correct number of grams

(c)

B1: Suggests a suitable refinement to the model

9. A company plans to build a new fairground ride. The ride will consist of a capsule that will hold the passengers and the capsule will be attached to a tall tower. The capsule is to be released from rest from a point half way up the tower and then made to oscillate in a vertical line.

The vertical displacement, x metres, of the top of the capsule below its initial position at time t seconds is modelled by the differential equation,

$$m\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 200\cos t, \qquad t \geqslant 0$$

where m is the mass of the capsule including its passengers, in thousands of kilograms.

The maximum permissible weight for the capsule, including its passengers, is 30 000 N.

Taking the value of g to be $10\,\text{ms}^{-2}$ and assuming the capsule is at its maximum permissible weight,

(a) (i) explain why the value of m is 3

(ii) show that a particular solution to the differential equation is

$$x = 40\sin t - 20\cos t$$

(iii) hence find the general solution of the differential equation.

(8)

(b) Using the model, find, to the nearest metre, the vertical distance of the top of the capsule from its initial position, 9 seconds after it is released.

(4)

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Question	Scheme	Marks	AOs
9(a)(i)	Weight = mass × g $\Rightarrow m = \frac{30\,000}{g} = 3000$ But mass is in thousands of kg, so $m = 3$	M1	3.3
(ii)	$\frac{dx}{dt} = 40\cos t + 20\sin t,\ \frac{d^2x}{dt^2} = -40\sin t + 20\cos t$	M1	1.1b
	$3(-40\sin t + 20\cos t) + 4(40\cos t + 20\sin t)$ $+40\sin t - 20\cos t = \ldots$	M1	1.1b
	$= 200\cos t$ so PI is $x = 40\sin t - 20\cos$	A1*	2.1
	or		
	Let $x = a\cos t + b\sin t$ $\frac{dx}{dt} = -a\sin t + b\cos t,\ \frac{d^2x}{dt^2} = -a\cos t - b\sin t$	M1	1.1b
	$4b - 2a = 200,\ -2b - 4a = 0 \Rightarrow a = \ldots, b = \ldots$	M1	2.1
	$x = 40\sin t - 20\cos t$	A1*	1.1b
(iii)	$3\lambda^2 + 4\lambda + 1 = 0 \Rightarrow \lambda = -1, -\frac{1}{3}$	M1	1.1b
	$x = Ae^{-t} + Be^{-\frac{1}{3}t}$	A1	1.1b
	$x = PI + CF$	M1	1.1b
	$x = Ae^{-t} + Be^{-\frac{1}{3}t} + 40\sin t - 20\cos t$	A1	1.1b
		(8)	
(b)	$t = 0, x = 0 \Rightarrow A + B = 20$	M1	3.4
	$x = 0, \frac{dx}{dt} = -Ae^{-t} - \frac{1}{3}Be^{-\frac{1}{3}t} + 40\cos t + 20\sin t = 0$ $\Rightarrow A + \frac{1}{3}B = 40$	M1	3.4
	$x = 50e^{-t} - 30e^{-\frac{1}{3}t} + 40\sin t - 20\cos t$	A1	1.1b
	$t = 9 \Rightarrow x = 33\text{m}$	A1	3.4
		(4)	
			(12 marks)

Question 9 notes:	
(a)(i)	
M1:	Correct explanation that in the model, $m = 3$
(ii)	
M1:	Differentiates the given PI twice
M1:	Substitutes into the given differential equation
A1*:	Reaches $200\cos t$ and makes a conclusion
or	
M1:	Uses the correct form for the PI and differentiates twice
M1:	Substitutes into the given differential equation and attempts to solve
A1*:	Correct PI
(iii)	
M1:	Uses the model to form and solve the auxiliary equation
A1:	Correct complementary function
M1:	Uses the correct notation for the general solution by combining PI and CF
A1:	Correct General Solution for the model
(b)	
M1:	Uses the initial conditions of the model, $t = 0$ at $x = 0$, to form an equation in A and B
M1:	Uses $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ at $x = 0$ in the model to form an equation in A and B
A1:	Correct PS
A1:	Obtains 33m using the assumptions made in the model

7. At the start of the year 2000, a survey began of the number of foxes and rabbits on an island.

At time t years after the survey began, the number of foxes, f, and the number of rabbits, r, on the island are modelled by the differential equations

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0.2f + 0.1r$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -0.2f + 0.4r$$

(a) Show that $\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 0.6\frac{\mathrm{d}f}{\mathrm{d}t} + 0.1f = 0$

(3)

(b) Find a general solution for the number of foxes on the island at time t years.

(4)

(c) Hence find a general solution for the number of rabbits on the island at time t years.

(3)

At the start of the year 2000 there were 6 foxes and 20 rabbits on the island.

(d) (i) According to this model, in which year are the rabbits predicted to die out?

(ii) According to this model, how many foxes will be on the island when the rabbits die out?

(iii) Use your answers to parts (i) and (ii) to comment on the model.

(7)

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Question	Scheme	Marks	AOs
7(a)	$r = 10\frac{\mathrm{d}f}{\mathrm{d}t} - 2f \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = 10\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 2\frac{\mathrm{d}f}{\mathrm{d}t}$	M1	2.1
	$10\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 2\frac{\mathrm{d}f}{\mathrm{d}t} = -0.2f + 0.4\left(10\frac{\mathrm{d}f}{\mathrm{d}t} - 2f\right)$	M1	2.1
	$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 0.6\frac{\mathrm{d}f}{\mathrm{d}t} + 0.1f = 0$*	A1*	1.1b
		(3)	
(b)	$m^2 - 0.6m + 0.1 = 0 \Rightarrow m = \frac{0.6 \pm \sqrt{0.6^2 - 4\times 0.1}}{2}$	M1	3.4
	$m = 0.3 \pm 0.1\mathrm{i}$	A1	1.1b
	$f = \mathrm{e}^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right)$	M1	3.4
	$f = \mathrm{e}^{0.3t}\left(A\cos 0.1t + B\sin 0.1t\right)$	A1	1.1b
		(4)	
(c)	$\frac{\mathrm{d}f}{\mathrm{d}t} = 0.3\mathrm{e}^{0.3t}\left(A\cos 0.1t + B\sin 0.1t\right) + 0.1\mathrm{e}^{0.3t}\left(B\cos 0.1t - A\sin 0.1t\right)$	M1	3.4
	$r = 10\frac{\mathrm{d}f}{\mathrm{d}t} - 2f$ $= \mathrm{e}^{0.3t}\left((3A+B)\cos 0.1t + (3B-A)\sin 0.1t\right) - 2\mathrm{e}^{0.3t}\left(A\cos 0.1t + B\sin 0.1t\right)$	M1	3.4
	$r = \mathrm{e}^{0.3t}\left((A+B)\cos 0.1t + (B-A)\sin 0.1t\right)$	A1	1.1b
		(3)	
(d)(i)	$t = 0, f = 6 \Rightarrow A = 6$	M1	3.1b
	$t = 0, r = 20 \Rightarrow B = 14$	M1	3.3
	$r = \mathrm{e}^{0.3t}\left(20\cos 0.1t + 8\sin 0.1t\right) = 0$	M1	3.1b
	$\tan 0.1t = -2.5$	A1	1.1b
	2019	A1	3.2a
(d)(ii)	3750 foxes	B1	3.4
(d)(iii)	e.g. the model predicts a large number of foxes are on the island when the rabbits have died out and this may not be sensible	B1	3.5a
		(7)	
			(17 marks)

Question 7 notes:	
(a)	
M1:	Attempts to differentiate the first equation with respect to t
M1:	Proceeds to the printed answer by substituting into the second equation
A1*:	Achieves the printed answer with no errors
(b)	
M1:	Uses the model to form and solve the auxiliary equation
A1:	Correct values for m
M1:	Uses the model to form the CF
A1:	Correct CF
(c)	
M1:	Differentiates the expression for the number of foxes
M1:	Uses this result to find an expression for the number of rabbits
A1:	Correct equation
(d)(i)	
M1:	Realises the need to use the initial conditions in the model for the number of foxes
M1:	Realises the need to use the initial conditions in the model for the number of rabbits to find both unknown constants
M1:	Obtains an expression for r in terms of t and sets = 0
A1:	Rearranges and obtains a correct value for tan
A1:	Identifies the correct year
(d)(ii)	
B1:	Correct number of foxes
(d)(iii)	
B1:	Makes a suitable comment on the outcome of the model

8. A large container initially contains 3 litres of pure water.
Contaminated water starts pouring into the container at a constant rate of 250 ml per minute and you may assume the contaminant dissolves completely.

At the same time, the container is drained at a constant rate of 125 ml per minute.
The water in the container is continually mixed.

The amount of contaminant in the water pouring into the container, at time t minutes after pouring began, is modelled to be $(5 - e^{-0.1t})$ mg per litre.

Let m be the amount of contaminant, in milligrams, in the container at time t minutes after the contaminated water begins pouring into the container.

(a) (i) Write down an expression for the total volume of water in litres in the container at time t.

(ii) Hence show that the amount of contaminant in the container can be modelled by the differential equation

$$\frac{dm}{dt} = \frac{5 - e^{-0.1t}}{4} - \frac{m}{24 + t}$$

(4)

(b) By solving the differential equation, find an expression for the amount of contaminant, in milligrams, in the container t minutes after the contaminated water begins to be poured into the container.

(8)

After 30 minutes, the concentration of contaminant in the water was measured as 3.79 mg per litre.

(c) Assess the model in light of this information, giving a reason for your answer.

(2)

Question	Scheme	Marks	AOs
8(a)(i)	Container contains $3+0.25t - 0.125t = 3 + 0.125t$ litres after t minutes	B1	3.3
(ii)	Rate of contaminant out $= 0.125 \times \dfrac{m}{3+0.125t}$ mg per minute	M1	3.3
	Rate of contaminant in $= 0.25 \times (5 - e^{-0.1t})$ mg per minute	B1	2.2a
	$\dfrac{dm}{dt} = \dfrac{5-e^{-0.1t}}{4} - \dfrac{m}{24+t}$ *	A1*	1.1b
		(4)	
(b)	Rearranges to form $\dfrac{dm}{dt} + \dfrac{m}{24+t} = \dfrac{5-e^{-0.1t}}{4}$ and attempts integrating factor (may be by recognition).	M1	3.1a
	I.F. $= \left(e^{\int \frac{1}{24+t} dt} = e^{\ln(24+t)} \right) = 24+t$	A1	1.1b
	$(24+t)m = \dfrac{1}{4}\int (24+t)(5-e^{-0.1t})\, dt = \dfrac{1}{4}\int 120 + 5t - 24e^{-0.1t} - te^{-0.1t}\, dt = ..$	M1	3.1a
	$= \dfrac{1}{4}\left(120t + \dfrac{5t^2}{2} - \dfrac{24e^{-0.1t}}{-0.1} + \ldots\right)$	A1	1.1b
	$\int te^{-0.1t}\, dt = t\dfrac{e^{-0.1t}}{-0.1} - \int 1 \times \dfrac{e^{-0.1t}}{-0.1}\, dt = t\dfrac{e^{-0.1t}}{-0.1} - \dfrac{e^{-0.1t}}{(-0.1)^2}$	M1 A1	1.1b 1.1b
	So $(24+t)m = \dfrac{5}{8}t^2 + 30t + 85e^{-0.1t} + \dfrac{5}{2}te^{-0.1t} + c$		
	When $t = 0$, $m = 0$ as initially no contaminant in the container, so $0 = 0 + 0 + 85 + 0 + c \Rightarrow c = -85$	M1	3.4
	$m = \dfrac{1}{24+t}\left(\dfrac{5}{8}t^2 + 30t + 85e^{-0.1t} + \dfrac{5}{2}te^{-0.1t} - 85\right)$	A1	2.2b
		(8)	
(c)	When $t = 30$ $m = 25.65677...$ and $V = 6.75$, hence the concentration is 3.80 mg per litre.	M1	3.4
	This resembles the measured value very closely and could easily be explained by minor inaccuracies in measurements, so the model seems to be suitable over this timeframe.	A1	3.5a
		(2)	
			(14 marks)

Notes:

(a)(i)

B1: A correct expression for the volume, may be unsimplified.

(ii)

M1: Expresses the amount of contaminant out in terms of m and t.

B1: Correct interpretation for amount of contaminant entering the container.

A1*: Puts all the components together to form the correct differential equation.

(b)

M1: Identifies the problem as a first order linear problem requiring integrating factor (by finding it or by recognition.

A1: Correct integrating factor

M1: Multiplies through by the IF, expands brackets on RHS and attempts the integration.

A1: Correct integration for first three terms.

M1: Integration by parts used on the $te^{-0.1t}$ term.

A1: Correct integration by parts.

M1: Uses the initial conditions to find the constant of integration – must have a constant of integration for this mark to be awarded.

A1: Correct expression for *m*, need not be simplified.

(c)

M1: Calculates the concentration from the model at $t = 30$

A1: Correct concentration found and uses it to make a comment on the validity of the model.

6. A damped spring is part of a car suspension system. In tests for the system, a mass is attached to the damped spring and is made to move upwards in a vertical line.

The motion of the system is modelled by the differential equation

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 2e^{-3t}$$

where x cm is the vertical displacement of the mass above its equilibrium position and t is the time, in seconds, after motion begins.

In one particular test, the mass is moved to a position 20 cm above its equilibrium position and given an initial velocity of 1 ms^{-1} upwards. For this test, use the model to

(a) find an equation for x in terms of t,

(9)

(b) find, to the nearest mm, the maximum displacement of the mass from its equilibrium position.

(3)

In this test, the time taken for the mass to return to its equilibrium position was measured as 2.86 seconds.

(c) State, with justification, whether or not this supports the model.

(1)

Question	Scheme	Marks	AOs
6(a)	$\frac{d^2x}{dt^2}+6\frac{dx}{dt}+9x=2e^{-3t}$		
	AE: $m^2 + 6m + 9 = 0 \Rightarrow (m+3)^2 = 0 \Rightarrow m = \ldots\ (= -3)$	M1	1.1b
	So C.F. is $x_{CF} = (A + Bt)e^{-3t}$	A1	2.2a
	For P.I. try $x_{PI} = kt^2e^{-3t}$	B1	2.2a
	$\dot{x}_{PI} = 2kte^{-3t} - 3kt^2e^{-3t}\quad (= k(2t-3t^2)e^{-3t})$ $\ddot{x}_{PI} = 2ke^{-3t} - 6kte^{-3t} - 6kte^{-3t} + 9kt^2e^{-3t}\quad (= k(2-12t+9t^2)e^{-3t})$ $\Rightarrow\ k(2-12t+9t^2)e^{-3t} + 6k(2t-3t^2)e^{-3t} + 9kt^2e^{-3t} = 2e^{-3t} \Rightarrow k = \ldots$	M1	1.1b
	So $k = 1$ ie $x_{PI} = t^2e^{-3t}$	A1	1.1b
	General solution is $x = (A + Bt)e^{-3t} + t^2e^{-3t}$ (their C.F. + their P.I.)	M1	1.1a
	$x(0) = 20 \Rightarrow A = 20$	M1	3.4
	$\dot{x} = Be^{-3t} - 3(A+Bt)e^{-3t} + 2te^{-3t} - 3t^2e^{-3t} = (B - 3A + (2-3B)t - 3t^2)e^{-3t}$ $\dot{x}(0) = 100 \Rightarrow B = 100 + 3A = \ldots (=160)$	M1	3.4
	So $x = (20 + 160t + t^2)e^{-3t}$	A1	1.1b
		(9)	
(b)	From above $\dot{x} = (B - 3A + (2-3B)t - 3t^2)e^{-3t} = (100 - 478t - 3t^2)e^{-3t}$		
	$\dot{x} = 0 \Rightarrow 100 - 478t - 3t^2 = 0 \Rightarrow t = ..\quad (= -159.5\ldots \text{ or } 0.2089\ldots)$	M1	3.1a
	$t > 0$, so $t_{max} = 0.2089\ldots \Rightarrow$ $x_{max} = \left(20 + 160\times 0.2089\ldots + (0.2089...)^2\right)e^{-3\times 0.2089\ldots} = \ldots$	M1	3.4
	$x_{max} =$ awrt 28.6 cm (3 s.f.) (28.57055381741878)	A1	1.1b
		(3)	
(c)	$x(2.86) = 0.0912\ldots$ which is close to zero (less than 1mm), which can be accounted for by inaccuracies in measurements. So the model is supported by this measurement.	B1ft	2.2b
		(1)	
		(13 marks)	

Notes:

(a)

M1: Forms and solves the auxiliary equation.

A1: Deduces correct C.F. for repeated root. (Variables must be consistent.)

B1: Deduces a correct form for the P.I. following a correct C.F. Accept any variations that include kt^2e^{-3t} with other terms.

M1: Differentiates their P.I. twice and substitutes into original equation and attempts to find the unknown(s).

A1: Correct value for k or correct P.I.

M1: Forms general solution, x = their C.F. + their P.I.

M1: Uses $x = 20$ at $t = 0$ to find first constant/set up one equation in two unknowns.

M1: Differentiates general solution and uses $\dot{x} = 100$ at $t = 0$ to form and solve second equation in the unknowns.

A1: Correct answer.

(b) **M1:** Uses $\dot{x} = 0$ to find the time the maximum is achieved. May use the derivative from (a) with constants found, or may differentiate again from answer to (a). **M1:** Substitutes t_{max} into their equations to find x_{max}. **A1:** Correct answer.
(c) **B1ft:** Finds x when t = 2.86 and makes an inference about whether it supports the model or not. The conclusion should be relevant for their found value, if close to zero then should conclude in accordance with model as may have slight variance due to measurements not being accurate, if not close to zero, then should conclude that even taking inaccuracies into account the measurement does not fit with the model.

5. A tank at a chemical plant has a capacity of 250 litres. The tank initially contains 100 litres of pure water.

Salt water enters the tank at a rate of 3 litres every minute. Each litre of salt water entering the tank contains 1 gram of salt.

It is assumed that the salt water mixes instantly with the contents of the tank upon entry.

At the instant when the salt water begins to enter the tank, a valve is opened at the bottom of the tank and the solution in the tank flows out at a rate of 2 litres per minute.

Given that there are S grams of salt in the tank after t minutes,

(a) show that the situation can be modelled by the differential equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 3 - \frac{2S}{100 + t}$$

(4)

(b) Hence find the number of grams of salt in the tank after 10 minutes.

(5)

When the concentration of salt in the tank reaches 0.9 grams per litre, the valve at the bottom of the tank must be closed.

(c) Find, to the nearest minute, when the valve would need to be closed.

(3)

(d) Evaluate the model.

(1)

Notes
M1: A complete strategy to find *A*, *B* and *C* e.g. partial fractions. Allow slip when finding the constant but must be the correct form of partial fractions and correct identity. M1: Starts the process of differences to identify the relevant fractions at the start and end. Must have attempted a minimum of $r=0,\quad r=1,\quad \ldots\quad r=n-1$ and $r=n$ Follow through on their values of *A*, *B* and C. Look for $r=0\rightarrow \frac{A}{1}-\frac{B}{2}+\frac{C}{3}$ $\qquad r=1\rightarrow \frac{A}{2}-\frac{B}{3}+\frac{C}{4}$ $r=n-1\rightarrow \frac{A}{n}-\frac{B}{n+1}+\frac{C}{n+2}$ $\qquad r=n\rightarrow \frac{A}{n+1}-\frac{B}{n+2}+\frac{C}{n+3}$ A1: Correct fractions from the beginning and end that do not cancel stated. M1 Combines all 'their' fractions (at least two algebraic fractions) over their correct common denominator, does not need to be the lowest common denominator (allow a slip in the numerator). A1: Correct answer. **Note**: if they start with $r=1$ the maximum they can score is M1M0A0M1A0 **Note:** Proof by induction gains no marks

Question	Scheme	Marks	AOs
5(a)	The tank initially contains 100L. 3 L are entering every minute and 2 L are leaving every minute so overall 1 L increase in volume each minute so the tank contains 100 + *t* litres after *t* minutes	M1	3.3
	2 L leave the tank each minute and if there are *S*g of salt in the tank, the concentration will be $\frac{S}{100+t}\,g/L$ so salt leaves the tank at a rate of $2\times\frac{S}{100+t}\,g$ per minute	M1	3.3
	Salt enters the tank at a rate of $3\times 1g$ per minute	B1	2.2a
	$\therefore \frac{\mathrm{d}S}{\mathrm{d}t}=3-\frac{2S}{100+t}$ * cso	A1*	1.1b
		(4)	
(b)	$\frac{\mathrm{d}S}{\mathrm{d}t}+\frac{2S}{100+t}=3$		
	$I=\mathrm{e}^{\int\frac{2}{100+t}\mathrm{d}t}=(100+t)^2\Rightarrow S(100+t)^2=\int 3(100+t)^2\mathrm{d}t$	M1	3.1b
	$S(100+t)^2=(100+t)^3\,(+c)$ OR $S(100+t)^2=30\,000t+300t^2+t^3\,(+c)$	A1	1.1b
	$t=0,\ S=0\Rightarrow c=-10^6$	M1	3.4
	$t=10\Rightarrow S=100+10-\frac{10^6}{(100+10)^2}$	dM1	1.1b

	OR $S(100+10)^2 = (100+10)^3(+c) \Rightarrow S = \ldots$		
	$=$ awrt 27 (g) or $\dfrac{3310}{121}$ (g)	A1	2.2b
		(5)	
(c)	Concentration is $\left(100+t-\dfrac{10^6}{(100+t)^2}\right) \div (100+t) = 0.9$ OR $S = 0.9\ 100+t \ \Rightarrow 0.9\ 100+t \ = \ 100+t \ - \dfrac{10^6}{100+10 \ ^2}$ OR $S = 0.9\ 100+t \ \Rightarrow 0.9\ 100+t \ ^3 = \ 100+t \ ^3 - 10^6$	M1	3.4
	$(100+t)^3 = 10^7 \Rightarrow t = \ldots$ OR $t^3 + 300t^2 + 30\,000t - 9\,000\,000 = 0 \Rightarrow t = \ldots$	dM1	1.1b
	$t =$ awrt 115 (minutes)	A1	2.2b
		(3)	
(d)	E.g. • It is unlikely that mixing is instantaneous • The model will only be valid when the tank is not full • When the valve is closed, the model is not valid • It is unlikely that the concentration of salt water entering the tank remains exactly the same	B1	3.5a
		(1)	
			(13 marks)

Notes

(a)

M1: A suitable explanation for the "100 + t" e.g. as a minimum (v) = 100 + 3t – 2t = 100 + t

M1: A suitable explanation for the $\dfrac{2S}{100+t}$

There need to be some explanation (words) for this part of the formula.

e.g. the concentration of (salt) $= \dfrac{S}{100+t}$ therefore (salt) out $= 2 \times \dfrac{S}{100+t} = \dfrac{2S}{100+t}$

e.g. salt out = $\dfrac{2S}{\text{volume of water}} = \dfrac{2S}{100+t}$

Note: M0 for $2 \times \dfrac{S}{100+t} = \dfrac{2S}{100+t}$ only with no explanation

B1: Correct interpretation for the "3" e.g. salt in = 3 or $\dfrac{dS}{dt}$ in = 3

<u>**Note:**</u> Salt **water** in = 3 is **B0**

A1*: Puts all the components together to form the given differential equation cso (b) M1: Uses the model to find the integrating factor and attempts the solution of the differential equation. Look for $I.F. = e^{\int \frac{2}{100+t} dt} \Rightarrow S \times \text{'their } I.F.\text{'} = \int 3 \times \text{'their } I.F.\text{' } dt$ A1: Correct solution condone missing + c **For the next three mark there must be a constant of integration** M1: Interprets the initial conditions, $t = 0 \ \ S = 0$, and uses in their equation to find the constant of integration. dM1: Dependent on having a constant of integration. Uses their solution to the problem to find the amount of salt after 10 minutes. A1: Awrt 27 or $\frac{3310}{121}$. (If the units are stated they must be correct) **<u>Note:</u>** If achieves $S(100+t)^2 = 30\,000t + 300t^2 + t^3 + c$ the constant of integration c = 0 and the correct amount of salt can be achieved. If there is no + c the maximum they can score is M1A1M0M0A0
Notes continued
(c) **Note:** Look out for setting S = 0.9 in this part, which scores no marks. M1: Uses their solution to the model and divides by 100 + t as an interpretation of the concentration and sets = 0.9. Alternatively recognises that the amount of salt = 0.9(100 + t) and substitutes for S in their solution to the model. dM1: Dependent on previous method mark. Solves their equation to obtain a value for t. May use a calculator. A1: Awrt 115 (If the units are stated they must be correct) or 1hr 45 mins with units (d) B1: Evaluates the model by making a suitable comment – see scheme for examples.

Question	Scheme	Marks	AOs
6	**<u>Way 1</u>** f(k +1) – f(k)		
	When n = 1, $3^{2n+4} - 2^{2n} = 729 - 4 = 725$ ($725 = 145 \times 5$) so the statement is true for n = 1	B1	2.2a
	Assume true for $n = k$ so $3^{2k+4} - 2^{2k}$ is divisible by 5	M1	2.4
	$f(k+1) - f(k) = 3^{2k+6} - 2^{2k+2} - 3^{2k+4} + 2^{2k}$	M1	2.1
	either $8f(k) + 5\times 2^{2k}$ or $3f(k) + 5\times 3^{2k+4}$	A1	1.1b
	$f(k+1) = 9f(k) + 5\times 2^{2k}$ or $f(k+1) = 4f(k) + 5\times 3^{2k+4}$ o.e.	A1	1.1b
	<u>If true for $n = k$ then it is true for</u>	A1	2.4

8. A scientist is studying the effect of introducing a population of white-clawed crayfish into a population of signal crayfish.
At time t years, the number of white-clawed crayfish, w, and the number of signal crayfish, s, are modelled by the differential equations

$$\frac{dw}{dt} = \frac{5}{2}(w - s)$$
$$\frac{ds}{dt} = \frac{2}{5}w - 90e^{-t}$$

(a) Show that

$$2\frac{d^2w}{dt^2} - 5\frac{dw}{dt} + 2w = 450e^{-t}$$

(3)

(b) Find a general solution for the number of white-clawed crayfish at time t years.

(6)

(c) Find a general solution for the number of signal crayfish at time t years.

(2)

The model predicts that, at time T years, the population of white-clawed crayfish will have died out.

Given that $w = 65$ and $s = 85$ when $t = 0$

(d) find the value of T, giving your answer to 3 decimal places.

(6)

(e) Suggest a limitation of the model.

(1)

Question	Scheme	Marks	AOs
8(a)	$\frac{\mathrm{d}^2w}{\mathrm{d}t^2}=\frac{5}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}t}-\frac{\mathrm{d}s}{\mathrm{d}t}\right)$ or $\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{\mathrm{d}w}{\mathrm{d}t}-\frac{2}{5}\frac{\mathrm{d}^2w}{\mathrm{d}t^2}$ o.e.	B1	1.1b
	$\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{\mathrm{d}w}{\mathrm{d}t}-\frac{2}{5}\frac{\mathrm{d}^2w}{\mathrm{d}t^2}\Rightarrow\frac{\mathrm{d}w}{\mathrm{d}t}-\frac{2}{5}\frac{\mathrm{d}^2w}{\mathrm{d}t^2}=\frac{2}{5}w-90\mathrm{e}^{-t}$	M1	2.1
	$2\frac{\mathrm{d}^2w}{\mathrm{d}t^2}-5\frac{\mathrm{d}w}{\mathrm{d}t}+2w=450\mathrm{e}^{-t}$ *	A1*	1.1b
		(3)	
(b)	$2m^2-5m+2=0\Rightarrow m=...$	M1	3.4
	$m=2,\frac{1}{2}$	A1	1.1b
	$(w)=A\mathrm{e}^{\alpha t}+B\mathrm{e}^{\beta t}$	M1	3.4
	$(w)=A\mathrm{e}^{0.5t}+B\mathrm{e}^{2t}$	A1	1.1b
	PI: Try $w=k\mathrm{e}^{-t}\Rightarrow\frac{\mathrm{d}w}{\mathrm{d}t}=-k\mathrm{e}^{-t}\Rightarrow\frac{\mathrm{d}^2w}{\mathrm{d}t^2}=k\mathrm{e}^{-t}$ $2k\mathrm{e}^{-t}+5k\mathrm{e}^{-t}+2k\mathrm{e}^{-t}=450\mathrm{e}^{-t}\Rightarrow k=...$	M1	3.4
	$w=$ 'their C.F.' $+\ 50\mathrm{e}^{-t}$ $\left(w=A\mathrm{e}^{0.5t}+B\mathrm{e}^{2t}+50\mathrm{e}^{-t}\right)$	A1ft	1.1b
		(6)	
(c)	$s=w-\frac{2}{5}\frac{\mathrm{d}w}{\mathrm{d}t}=A\mathrm{e}^{0.5t}+B\mathrm{e}^{2t}+50\mathrm{e}^{-t}-\frac{2}{5}\left(\frac{A}{2}\mathrm{e}^{0.5t}+2B\mathrm{e}^{2t}-50\mathrm{e}^{-t}\right)$	M1	3.4
	$s=\frac{4A}{5}\mathrm{e}^{0.5t}+\frac{B}{5}\mathrm{e}^{2t}+70\mathrm{e}^{-t}$	A1	1.1b
		(2)	
(d)	$65=A+B+50,\ 85=\frac{4A}{5}+\frac{B}{5}+70\Rightarrow A=...,B=...$ $(\text{NB } A=20\ \ B=-5)$	M1	3.3
	$w=0\Rightarrow 20\mathrm{e}^{0.5t}-5\mathrm{e}^{2t}+50\mathrm{e}^{-t}=0$	dM1	1.1b
	$\mathrm{e}^{3t}-4\mathrm{e}^{1.5t}-10(=0)$ or a multiple	A1	3.1a
	$\mathrm{e}^{1.5t}=\frac{4\pm\sqrt{4^2-4\times(1)(-10)}}{2}$	M1	1.1b
	$1.5t=\ln\left(\frac{4+\sqrt{56}}{2}\right)$	M1	2.3
	$T=\frac{2}{3}\ln\left(\frac{4+\sqrt{56}}{2}\right)=$ awrt 1.165	A1	3.2a
		(6)	

(e)	E.g. • Either population becomes negative which is not possible • When the white-clawed crayfish have died out, the model will not be valid	B1	3.5b
		(1)	
			(18 marks)

Notes

(a)

B1: Differentiates the first equation with respect to t correctly.

M1: Substitutes $\frac{\mathrm{d}s}{\mathrm{d}t}$ into their derivative.

A1*: Achieves the printed answer with no errors.

(b) **<u>Note:</u> All the mark except the final A1 are available if they use other variables.**

M1: Uses the model to form and solve the Auxiliary Equation.

A1: Correct roots of the AE.

M1: Uses the model to form the Complementary Function for their roots (they may be complex roots)

A1: Correct CF

M1: Chooses the correct form of the PI according to the model and uses a complete method to find the PI. Uses $w = ke^{-t}$ finds both $\frac{\mathrm{d}w}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2w}{\mathrm{d}t^2}$ substitutes into the differential equation and find the value of k.

A1ft: Dependent on all three of the previous method marks. Following through on their CF only to give $w = \text{'their CF'} + 50e^{-t}$

(c)

M1: Substitutes into the first equation the answer for part (b) in place of w and the derivative of their (b) in place of $\frac{\mathrm{d}w}{\mathrm{d}t}$. If they rearrange to make S the subject first and make a slip but still substitutes for w and $\frac{\mathrm{d}w}{\mathrm{d}t}$ allow this mark.

A1: Correct simplified equation.

(d)

M1: Uses the initial conditions $t = 0, w = 65$ and $s = 85$ to form simulations equations and solves to find the values of their constants

dM1: Dependent on the previous method mark. Sets $w = 0$

A1: Processes the indices correctly to obtain a 3-term quadratic equation in terms of $e^{1.5t}$. It does not need to all be on one side and condone missing = 0.

M1: Solves their three-term quadratic (3TQ) to reach $e^{pt} = q$

M1: Correct use of logarithms to reach $pt = \ln q$ where $q > 0$ and rejects the other solution

A1: awrt 1.165

Note: the final 3 marks only can be implied by a correct answer following the correct 3-term quadratic equation in terms of $e^{1.5t}$

(e)
B1: Suggests a suitable limitation of the model, not valid when negative population
Any mention of other factors such as does not take into account e.g. other predictors, fishing, disease, lack of food etc is B0

5. Two compounds, X and Y, are involved in a chemical reaction. The amounts in grams of these compounds, t minutes after the reaction starts, are x and y respectively and are modelled by the differential equations

$$\frac{dx}{dt} = -5x + 10y - 30$$

$$\frac{dy}{dt} = -2x + 3y - 4$$

(a) Show that

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 50$$

(3)

(b) Find, according to the model, a general solution for the amount in grams of compound X present at time t minutes.

(6)

(c) Find, according to the model, a general solution for the amount in grams of compound Y present at time t minutes.

(3)

Given that $x = 2$ and $y = 5$ when $t = 0$

(d) find

(i) the particular solution for x,

(ii) the particular solution for y.

(4)

A scientist thinks that the chemical reaction will have stopped after 8 minutes.

(e) Explain whether this is supported by the model.

(1)

Question	Scheme	Marks	AOs
5(a)	$\frac{d^2x}{dt^2}=-5\frac{dx}{dt}+10\frac{dy}{dt}$ oe e.g. $\frac{dy}{dt}=\frac{1}{10}\left(\frac{d^2x}{dt^2}+5\frac{dx}{dt}\right)$	B1	1.1b
	$\frac{d^2x}{dt^2}=-5\frac{dx}{dt}+10(-2x+3y-4)$ $=-5\frac{dx}{dt}-20x+\frac{30}{10}\left(\frac{dx}{dt}+5x+30\right)-40$ Or $\frac{1}{10}\left(\frac{d^2x}{dt^2}+5\frac{dx}{dt}\right)=-2x+\frac{3}{10}\left(30+5x+\frac{dx}{dt}\right)-4$	M1	2.1
	$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=50$ *	A1*	1.1b
		(3)	
(b)	$m^2+2m+5=0\Rightarrow m=\ldots$	M1	3.4
	$m=-1\pm 2\mathrm{i}$	A1	1.1b
	$m=\alpha\pm\beta\mathrm{i}\Rightarrow x=e^{\alpha t}(A\cos\beta t+B\sin\beta t)=\ldots$	M1	3.4
	$x=e^{-t}(A\cos 2t+B\sin 2t)$	A1	1.1b
	PI: Try $x=k\Rightarrow 5k=50\Rightarrow k=10$	M1	3.4
	$GS: x=e^{-t}(A\cos 2t+B\sin 2t)+10$	A1ft	1.1b
		(6)	
(c)	$\frac{dx}{dt}=e^{-t}(2B\cos 2t-2A\sin 2t)-e^{-t}(A\cos 2t+B\sin 2t)$	B1ft	1.1b
	$(y=)\frac{1}{10}\left(\frac{dx}{dt}+5x+30\right)=\ldots$	M1	3.4
	$y=\frac{1}{10}e^{-t}\left((4A+2B)\cos 2t+(4B-2A)\sin 2t\right)+8$	A1	1.1b
		(3)	
(d)	$t=0, x=2\Rightarrow 2=A+10\Rightarrow A=-8$	M1	3.1b
	$t=0, y=5\Rightarrow 5=\frac{1}{10}(2B-32)+8\Rightarrow B=1$	M1	3.3
	$x=e^{-t}(\sin 2t-8\cos 2t)+10$	A1	2.2a
	$y=e^{-t}(2\sin 2t-3\cos 2t)+8$	A1	2.2a
		(4)	
(e)	E.g When t > 8, the amount of compound X and the amount of compound Y remain (approximately) constant at 10 and 8 respectively, which suggests that the chemical reaction has stopped. This supports the scientist's claim.	B1	3.5a
		(1)	
			(17 marks)

Notes
(a) B1: Differentiates the first equation with respect to t correctly. May have rearranged to make y the subject first. The dot notation for derivatives may be used. M1: Uses the second equation to eliminate y to achieve an equation in x, $\frac{dx}{dt}, \frac{d^2x}{dt^2}$. A1*: Achieves the printed answer with no errors. (b) M1: Uses the model to form and attempts to attempts to solve the auxiliary equation (Accept a correct equation followed by two values for m as an attempt to solve.) A1: Correct roots of the AE M1: Uses the model to form the complementary function. Must be in terms of t only (not x) A1: Correct CF M1: Chooses the correct form of the PI according to the model and uses a complete method to find the PI A1ft: Combines their CF (which need not be correct) with the correct PI to give x in terms of t so look for x = their CF + 10 (c) B1ft: Correct differentiation of their x. Follow through their $e^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right)$ M1: Uses the model and their answer to part (b) to find an expression for y in terms of t A1: Correct equation. Mark the final answer but there is not need for terms to be gathered but must have $y = \ldots$. (d) M1: Realises the need to use the initial conditions in the equation for x M1: Realises the need to use the initial conditions in the equation for y to find both unknown constants - must have equations from which both unknowns can be found. Alternatively, a complete method using $\frac{dx}{dt}$ to find the second constant is made. A1: Deduces the correct equation for x A1: Deduces the correct equation for y. For this equation constants should have been gathered. (e) B1: Allow for any appropriate comment with valid supporting reason. They must have equations of the correct form from (d). The coefficients may be incorrect, but they must have positive limits for each of x and y. Both x and y should be considered (see below for exception), and a reason and some comment about the suitability of the model made (though you may allow implicit conclusions). E.g. • for values of $t > 8$, the amounts of compounds X and Y present settle at 10 and 8 without really varying, which supports the claim. • $\frac{dx}{dt} \approx 0$ and $\frac{dy}{dt} \approx 0$ when $t = 8$, so neither are changing, which supports the claim. • As t gets large x and y tend to limits to 10 and 8 neither will be zero, hence the claim is not supported. • $x = 10.0$ (awrt) and $y = 8.00$ (awrt) when $t = 8$, since neither is zero it is likely the reaction is still continuing so the claim is not supported. Exception: Allow a reason that states the model assumes that the reaction continues indefinitely, so the claim is not supported. (The reaction stopping would require a change in the model.)

Do NOT allow an answer that only considers x or y. E.g. $x = 10$ when $t = 8$ so the model is not supported is B0 since there is no consideration that y may be zero and hence end the reaction.

Alt for (c) and (d) restarting:

B1: Correct second order equation for y formed: $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 5y = 40$

M1: Full method to obtain the general solution: they may recognise the similarity to the equation in x and jump straight to finding the PI, or may form the aux equation etc again, but look for an attempt that combines a (correctly formed) CF and a PI. For this mark allow if the constants used are the same as those for the equation in x.

A1: Correct solution for y with different constants than those for x, though allow recovery if they realise in (d) that they need different constants.

For (d)

M1: As main scheme, allow for using the initial conditions in one equation to make a start finding the constants.

M1: For a full method to obtain all four constants – if the same constant were used for both equations in (c) (inconsistently) then this mark cannot be scored. A full method here would, for instance, require finding $\frac{\mathrm{d}x}{\mathrm{d}t}$ and using this along with the given initial equations and initial conditions to find the second constants for each equation.

A1: One correct equation with SC of being qualified by the first M only if a full method to find both constants for just one equation is made (so M1M0A1A0 is possible in this case).

A1: Both equations correct.

3. A scientist is investigating the concentration of antibodies in the bloodstream of a patient following a vaccination.
The concentration of antibodies, x, measured in micrograms (μg) per millilitre (ml) of blood, is modelled by the differential equation

$$100\frac{d^2x}{dt^2} + 60\frac{dx}{dt} + 13x = 26$$

where t is the number of weeks since the vaccination was given.

(a) Find a general solution of the differential equation.

(4)

Initially,

- there are no antibodies in the bloodstream of the patient
- the concentration of antibodies is estimated to be increasing at 10 μg/ml per week

(b) Find, according to the model, the maximum concentration of antibodies in the bloodstream of the patient after the vaccination.

(8)

A second dose of the vaccine has to be given to try to ensure that it is fully effective.
It is only safe to give the second dose if the concentration of antibodies in the bloodstream of the patient is less than 5 μg/ml.

(c) Determine whether, according to the model, it is safe to give the second dose of the vaccine to the patient exactly 10 weeks after the first dose.

(2)

Question	Scheme	Marks	AOs
3(a)	$100m^2+60m+13=0 \Rightarrow m=-0.3\pm 0.2\mathrm{i}$	M1	1.1b
	$x=\mathrm{e}^{-0.3t}\left(A\cos 0.2t+B\sin 0.2t\right)$	A1	1.1b
	PI: $x = 2$	B1	1.1b
	$x=\mathrm{e}^{-0.3t}\left(A\cos 0.2t+B\sin 0.2t\right)+2$	A1ft	2.2a
		(4)	
(b)	$t=0,\ \ x=0 \Rightarrow A=-2$	M1	3.4
	$\dfrac{\mathrm{d}x}{\mathrm{d}t}=-0.3\mathrm{e}^{-0.3t}\left(-2\cos 0.2t+B\sin 0.2t\right)+\mathrm{e}^{-0.3t}\left(0.4\sin 0.2t+0.2B\cos 0.2t\right)$ $t=0,\ \dfrac{\mathrm{d}x}{\mathrm{d}t}=10 \Rightarrow B=\ldots$(NB B = 47)	M1	3.4
	$x=\mathrm{e}^{-0.3t}\left(47\sin 0.2t-2\cos 0.2t\right)+2$	A1	1.1b
	$-0.3\mathrm{e}^{-0.3t}\left(47\sin 0.2t-2\cos 0.2t\right)+\mathrm{e}^{-0.3t}\left(9.4\cos 0.2t+0.4\sin 0.2t\right)=0$ $\Rightarrow t=\ldots$ or $x=\sqrt{2213}\mathrm{e}^{-0.3t}\sin\left(0.2t-0.0425\right)+2$ $\Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}t}=-0.3\sqrt{2213}\mathrm{e}^{-0.3t}\sin\left(0.2t-0.0425\right)$ $+0.2\sqrt{2213}\mathrm{e}^{-0.3t}\cos\left(0.2t-0.0425\right)$ $\Rightarrow t=\ldots$	M1	3.1b
	$\tan 0.2t=\dfrac{100}{137} \Rightarrow 0.2t=0.630\ldots$ or $\tan\left(0.2t-0.0425\right)=\dfrac{2}{3} \Rightarrow 0.2t=0.630$	M1	2.1
	t = 3.15… weeks	A1	1.1b
	$x=\mathrm{e}^{-0.3\times\text{"3.15..."}}\left(47\sin\left(0.2\times\text{"3.15..."}\right)-2\cos\left(0.2\times\text{"3.15..."}\right)\right)+2$	M1	3.4
	= awrt 12.1 {μg/ml}	A1	3.2a
		(8)	
(c)	$t=10 \Rightarrow x=\mathrm{e}^{-3}\left(47\sin(2)-2\cos(2)\right)+2=4.16\ldots$	M1	3.4
	The model suggests that it would be safe to give the second dose	A1ft	2.2a
		(2)	
			(14 marks)

Notes

(a)
M1: Uses the model to form and solve the auxiliary equation
A1: Correct CF, does not need x =
B1: Correct PI
A1ft: Deduces the correct GS (follow through their CF + PI). Must have $x = \mathrm{f}(t)$ and PI not 0

(b)
M1: Uses the model and the initial conditions to establish the value of "A"
M1: Differentiates their model using the product rule and uses the initial conditions to establish the value of "B". Must be using $x = 0$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 10$
A1: Correct particular solution. This can be implied by the correct constants found following a correct answer to part (a).

M1: Uses their solution to the model with a correct strategy to obtain the required value of t e.g. differentiates, sets equal to zero and solves for t
M1: Uses a correct trigonometric approach that leads to a value for t
A1: Correct value for t
M1: Uses the model and their value for t to find the maximum concentration.
A1: Correct value
(c)
M1: Uses the model to find the concentration when $t = 10$
A1ft: Makes a suitable comment that is consistent with their calculated value
Special case: If the candidate's maximum value is less than 5 then
M1: never reaches 5 as maximum is…. or max is less than 5
A1: yes, it is safe

6. A tourist decides to do a bungee jump from a bridge over a river.
One end of an elastic rope is attached to the bridge and the other end of the elastic rope is attached to the tourist.
The tourist jumps off the bridge.

At time t seconds after the tourist reaches their lowest point, their vertical displacement is x metres above a fixed point 30 metres vertically above the river.

When $t = 0$

- $x = -20$
- the velocity of the tourist is $0\,\text{m}\,\text{s}^{-1}$
- the acceleration of the tourist is $13.6\,\text{m}\,\text{s}^{-2}$

In the subsequent motion, the elastic rope is assumed to remain taut so that the vertical displacement of the tourist can be modelled by the differential equation

$$5k\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}x}{\mathrm{d}t} + 17x = 0 \qquad t \geqslant 0$$

where k is a positive constant.

(a) Determine the value of k

(2)

(b) Determine the particular solution to the differential equation.

(7)

(c) Hence find, according to the model, the vertical height of the tourist above the river 15 seconds after they have reached their lowest point.

(2)

(d) Give a limitation of the model.

(1)

Question	Scheme	Marks	AOs
6(a)	$5k(13.6)+2k(0)+17(-20)=0 \Rightarrow k=...$	M1	3.3
	$k=5$	A1	1.1b
		(2)	
(b)	Solves their $25m^2+10m+17=0 \Rightarrow m=...$	M1	3.1b
	$m=-0.2\pm0.8\text{i}$	A1	1.1b
	$x=\text{e}^{-0.2t}(A\cos0.8t+B\sin0.8t)$	A1ft	1.1b
	$t=0,\ x=-20 \Rightarrow A=...\ (=-20)$	M1	3.4
	$\frac{\text{d}x}{\text{d}t}=-0.2\text{e}^{-0.2t}(A\cos0.8t+B\sin0.8t)$ $+\text{e}^{-0.2t}(-0.8A\sin0.8t+0.8B\cos0.8t)$	M1	1.1b
	$t=0\ \frac{\text{d}x}{\text{d}t}=0 \Rightarrow -0.2A+0.8B=0 \Rightarrow B=...\ (=-5)$	dM1	3.4
	$x=\text{e}^{-0.2t}(-20\cos0.8t-5\sin0.8t)$ o.e.	A1	1.1b
		(7)	
(c)	Vertical height $=30+\left[\text{e}^{-0.2\times15}(-20\cos(0.8\times15)-5\sin(0.8\times15))\right]$	M1	3.4
	Vertical height = awrt 29.3 m	A1	2.2b
		(2)	
(d)	For example It is unlikely that the rope will remain taut The model predicts the tourist will continue to move up and down, (but in fact they will lose momentum) The tourist is modelled as a particle	B1	3.5b
		(1)	
			(12 marks)

Notes:

(a)

M1: Substitutes $\frac{\text{d}^2x}{\text{d}t^2}=13.6$ $\frac{\text{d}x}{\text{d}t}=0$ and $x=-20$ into the differential equation to find a value for k.
Allow if there are sign slips but must be attempting the values in the correct places.

A1: Correct value $k=5$

(b)

M1: Forms and solves the auxiliary equation.

A1: Correct solution to the auxiliary equation (not follow through).

A1ft: Correct complementary function for their solutions to their auxiliary equation. (Follow through on distinct real, repeated or complex roots.)

M1: Uses the information from the model $t = 0$ $x = -20$ to find a constant or equation linking two constants in their equation. **M1:** Differentiates an expression of the form $e^{kt}\left(A\cos\lambda_1 t + B\sin\lambda_2 t\right)$ using the product rule to find an expression for the velocity. **dM1:** Uses the information from the model, $t = 0$ $\frac{dx}{dt} = 0$ to find and solve another equation for the constants. **A1:** Correct equation for displacement.
(c) **M1:** Finds the height above the river by finding the displacement after 15 seconds and adding 30 **A1**: Vertical height = awrt 29.3 m
(d) **B1:** Any suitable comment relating to the given model or the outcomes of it. See scheme for examples. Do not accept just "air resistance has not been considered" as the question does not say this was ignored. However, if a valid consequence of what including air resistance would mean to the model, then the mark may be awarded.

8. Two different colours of paint are being mixed together in a container.

The paint is stirred continuously so that each colour is instantly dispersed evenly throughout the container.

Initially the container holds a mixture of 10 litres of red paint and 20 litres of blue paint.

The colour of the paint mixture is now altered by

- adding red paint to the container at a rate of 2 litres per second
- adding blue paint to the container at a rate of 1 litre per second
- pumping fully mixed paint from the container at a rate of 3 litres per second.

Let r litres be the amount of red paint in the container at time t seconds after the colour of the paint mixture starts to be altered.

(a) Show that the amount of red paint in the container can be modelled by the differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 2 - \frac{r}{\alpha}$$

where α is a positive constant to be determined.

(2)

(b) By solving the differential equation, determine how long it will take for the mixture of paint in the container to consist of equal amounts of red paint and blue paint, according to the model. Give your answer to the nearest second.

(6)

It actually takes 9 seconds for the mixture of paint in the container to consist of equal amounts of red paint and blue paint.

(c) Use this information to evaluate the model, giving a reason for your answer.

(1)

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question	Scheme		Marks	AOs
8(a)	Volume of paint = 30 litres therefore Rate of paint out $= 3 \times \dfrac{r}{30}$ litres per second		M1	3.3
	$\dfrac{\mathrm{d}r}{\mathrm{d}t} = 2 - \dfrac{r}{10}$		A1	1.1b
			(2)	
(b)	Rearranges $\dfrac{\mathrm{d}r}{\mathrm{d}t} + \dfrac{r}{10} = 2$ and attempts integrating factor IF = $\mathrm{e}^{\int \frac{1}{10}\mathrm{d}t} = \ldots$	Separates the variables $\int \dfrac{1}{20-r}\,\mathrm{d}r = \dfrac{1}{10}\mathrm{d}t$ $\Rightarrow \ldots$	M1	3.1a
	$r\mathrm{e}^{\frac{t}{10}} = \int 2\mathrm{e}^{\frac{t}{10}}\,\mathrm{d}t \Rightarrow r\mathrm{e}^{\frac{t}{10}} = \lambda \mathrm{e}^{\frac{t}{10}}(+c)$	Integrates to the form $\lambda \ln(20-r) = \dfrac{1}{10}t(+c)$	M1	1.1b
	$r\mathrm{e}^{\frac{t}{10}} = 20\mathrm{e}^{\frac{t}{10}} + c$	$-\ln(20-r) = \dfrac{1}{10}t + c$	A1ft	1.1b
	$t = 0,\ r = 10 \Rightarrow c = \ldots$		M1	3.4
	$r = \dfrac{20\mathrm{e}^{\frac{t}{10}} - 10}{\mathrm{e}^{\frac{t}{10}}} = 15$ rearranges to achieve $\mathrm{e}^{\frac{t}{10}} = \alpha$ and solves to find a value for t or $r = 20 - 10\mathrm{e}^{-\frac{t}{10}} = 15$ rearranges to achieve $\mathrm{e}^{-\frac{t}{10}} = \beta$ and solves to find a value for t	$-\ln(20-15) = \dfrac{1}{10}t - \ln 10$ Leading to a value for t	M1	3.4
	$t =$ awrt 7 seconds		A1	2.2b
			(6)	
(c)	The model predicts 7 seconds but it actually takes 9 seconds so (over) 2 seconds out (over 20%), therefore it is not a good model		B1ft	3.5a
			(1)	
				(9 marks)

Notes:

(a)

M1: Clearly identifies that Rate of paint out $= 3 \times \dfrac{r}{\text{their volume}}$. It is a "show that" question so there must be clearly reasoning. Just answer with no reasoning scores M0.

A1: Puts all the components together to form the correct differential equation.

(b) **M1:** Identifies as a first order differential equation and finds the integrating factor or separates the variables and integrates. Allow if there are sign slips in rearranging (e.g. to $\frac{\mathrm{d}r}{\mathrm{d}t}-\frac{r}{10}=2$) or in the integrating factor and allow with their value for *a* or with *a* as an unknown. **M1:** Multiplies through by the IF and attempts to integrate or integrates to the form $\lambda\ln(2a-r)=\frac{1}{a}t+c$ oe **A1ft:** Correct integration, including constant of integration. Follow through on their value of *a*, but not sign slips from rearrangement. So allow for $r\mathrm{e}^{\frac{t}{a}}=2a\,\mathrm{e}^{\frac{t}{a}}+c$ or $-\ln(2a-r)=\frac{1}{a}t+c$ oe with *a* or their *a*. **M1:** Uses the initial conditions to find the constant of integration. Must see substitution or can be implied by the correct value for their equation. Allow for finding in terms of *a* if separation of variables used. **M1:** Sets $r=15$, achieves $\mathrm{e}^{\frac{t}{10}}=\alpha>0$ or $\mathrm{e}^{-\frac{t}{10}}=\beta>0$ as appropriate and solves to find a value for *t*. Separates the variable method sets *r* = 15 and rearranges to find a value for *t*. **Note:** For this mark a value of *a* is needed, but need not be the correct one. **A1cso:** $t=\text{awrt }7$ seconds from fully correct work.
(c) **B1ft:** See scheme, follow through on their answer to part (b). Accept any reasonable comparative comment but must have a reason, not just a statement of good or not good. So e.g. look for finding the difference between their answer and 9, or the percentage difference. If their answer is close to 9, then accept a conclusion of being a good model if a suitable reason is given. May substitute 9 into their equation and obtain a value to compare with 15 and make a similar conclusion.

10.

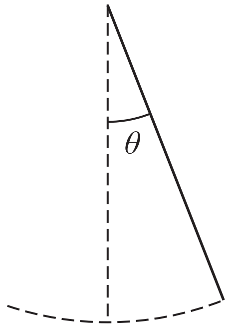

Figure 3

The motion of a pendulum, shown in Figure 3, is modelled by the differential equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 9\theta = \frac{1}{2}\cos 3t$$

where θ is the angle, in radians, that the pendulum makes with the downward vertical, t seconds after it begins to move.

(a) (i) Show that a particular solution of the differential equation is

$$\theta = \frac{1}{12}t\sin 3t$$

(4)

(ii) Hence, find the general solution of the differential equation.

(4)

Initially, the pendulum

- makes an angle of $\frac{\pi}{3}$ radians with the downward vertical
- is at rest

Given that, 10 seconds after it begins to move, the pendulum makes an angle of α radians with the downward vertical,

(b) determine, according to the model, the value of α to 3 significant figures.

(4)

Given that the true value of α is 0.62

(c) evaluate the model.

(1)

The differential equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 9\theta = \frac{1}{2}\cos 3t$$

models the motion of the pendulum as moving with forced harmonic motion.

(d) Refine the differential equation so that the motion of the pendulum is simple harmonic motion.

(1)

uestion	Scheme		Marks	AOs
10(a)(i)	$\frac{d\theta}{dt} = \alpha \sin 3t + \beta t \cos 3t$ and $\frac{d^2\theta}{dt^2} = \delta \cos 3t + \gamma t \sin 3t$	Let $\theta = \lambda t \sin 3t$ $\frac{d\theta}{dt} = \alpha \sin 3t + \beta t \cos 3t$ and $\frac{d^2\theta}{dt^2} = \delta \cos 3t + \gamma t \sin 3t$	M1	1.1b
	$\frac{d\theta}{dt} = \frac{1}{12}\sin 3t + \frac{1}{4}t\cos 3t$ and $\frac{d^2\theta}{dt^2} = \frac{1}{4}\cos 3t + \frac{1}{4}\cos 3t - \frac{3}{4}t\sin 3t$ $= \frac{1}{2}\cos 3t - \frac{3}{4}t\sin 3t$	$\frac{d\theta}{dt} = \lambda \sin 3t + 3\lambda t\cos 3t$ and $\frac{d^2\theta}{dt^2} = 3\lambda\cos 3t + 3\lambda\cos 3t - 9\lambda t\sin 3t$ $= 6\lambda\cos 3t - 9\lambda t\sin 3t$	A1	1.1b
	$\frac{1}{2}\cos 3t - \frac{3}{4}t\sin 3t + 9\left(\frac{1}{12}t\sin 3t\right) = \ldots$	$6\lambda\cos 3t - 9\lambda t\sin 3t + 9(\lambda t\sin 3t) = \frac{1}{2}\cos 3t \Rightarrow \lambda = \ldots$	dM1	3.4
	$= \frac{1}{2}\cos 3t$ so PI is $\theta = \frac{1}{12}t\sin 3t$ *	$\theta = \frac{1}{12}t\sin 3t$ *	A1*	2.1
			(4)	
(a)(ii)	$m^2 + 9 = 0 \Rightarrow m = \pm 3i$		M1	1.1b
	$\theta = A\cos 3t + B\sin 3t$		A1	1.1b
	$(\theta =)CF + PI$		dM1	1.1b
	$\theta = A\cos 3t + B\sin 3t + \frac{1}{12}t\sin 3t$		A1	1.1b
			(4)	
(b)	$t = 0,\ \theta = \frac{\pi}{3} \Rightarrow A = \ldots\left\{\frac{\pi}{3}\right\}$		M1	3.4
	$t = 0,\ \frac{d\theta}{dt} = -3A\sin 3t + 3B\cos 3t + \frac{1}{12}\sin 3t + \frac{1}{4}t\cos 3t = 0$ $\Rightarrow B = \ldots\{0\}$		M1	3.4
	$\alpha = \frac{\pi}{3}\cos(3\times 10) + \frac{1}{12}(10)\sin(3\times 10) = \ldots$		ddM1	1.1b
	$\alpha = \pm$awrt 0.662		A1	3.4
			(4)	
(c)	0.662 is close to 0.62 so a good model (at $t = 10$)		B1ft	3.5a
			(1)	
(d)	$\frac{d^2\theta}{dt^2} + 9\theta = 0$ oe		B1	3.5c
			(1)	

(14 marks)
Notes:
(a)(i) Note: mark (a) as a whole **M1:** Differentiates the given PI twice using the product rule to achieve the required form. Alternatively, uses a correct form for the PI and differentiates twice using the product rule to achieve the required form. A correct form may involve other terms with coefficients that will be zero, e.g. $\theta = \lambda t \sin 3t + \mu t \cos 3t$ is fine. Also allow e.g $\theta = \lambda t \sin \quad \omega t$ **A1:** Correct derivatives. **dM1:** Depends on first M, substitutes into the given differential equation and attempts to simplify. In the Alt they must go on to find value for λ . **A1*:** Achieves $\frac{1}{2}\cos 3t$and makes a minimal conclusion (e.g //). Alternatively reaches the correct PI.
(a)(ii) **M1:** Uses the model to form and solve the auxiliary equation. Accept $m^2 + 9 = 0 \rightarrow m = \pm 3i$ or $\pm$ 3 **A1**: Correct complementary function. Must be in terms of *t* but allow recovery if initially in terms of *x* but changed later. **dM1:** Dependent on the previous method mark. Finds the general solution by adding the particular integral to the complementary function. **A1:** Correct general solution including "$\theta =$ ", which may be recovered in part (b).
(b) **M1:** Uses the initial conditions of the model, $t = 0,\ \theta = \frac{\pi}{3}$to find a value for a constant. **M1:** Differentiates the general solution and uses the initial conditions of the model $t = 0,\ \frac{d\theta}{dt} = 0$ to find a value for the other constant. **ddM1:** Dependent on both previous method marks. Substitutes *t* = 10 into their particular solution. If not substitution is seen, accept any value as the attempt as long as they have found all relevant constants. **A1:** Accept awrt ±0.662
(c) **B1ft:** Makes a quantitative comparison of the size of their answer to part (b) with 0.62 and makes conclusion (e.g. good model). Follow through on their answer to (b) and draws an appropriate conclusion about the model. Accept “not reasonable” as long as it is supported with evidence but there must be some instructive comparison and a conclusion about the model - not just stating how much it is out. The reason given must be correct. Accept e.g. a correct percentage error with reasonable conclusion, or statement approximately equal with conclusion. Do not accept e.g. “does not agree to 1 s.f.” or “out by 0.6” as these lacks context. Do not accept arguments based solely on a difference in sign, they must be referring to the relative size of angle.
(d) **B1:** Refines the model, accept any constant on the right hand side.