Cp2Ch7 XMQs and MS

(Total: 29 marks)

1. CP2_2019   Q5 . 12 marks - CP2ch7 Methods in differential equations
2. CP1_2020   Q7 . 11 marks - CP2ch7 Methods in differential equations
3. CP1_2022   Q3 . 6 marks - CP2ch7 Methods in differential equations

5. An engineer is investigating the motion of a sprung diving board at a swimming pool.
Let E be the position of the end of the diving board when it is at rest in its equilibrium position and when there is no diver standing on the diving board.
A diver jumps from the diving board.
The vertical displacement, h cm, of the end of the diving board above E is modelled by the differential equation

$$4\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + 4\frac{\mathrm{d}h}{\mathrm{d}t} + 37h = 0$$

where t seconds is the time after the diver jumps.

(a) Find a general solution of the differential equation.
(2)

When $t = 0$, the end of the diving board is 20 cm below E and is moving upwards with a speed of $55\,\mathrm{cm\,s^{-1}}$.

(b) Find, according to the model, the maximum vertical displacement of the end of the diving board above E.
(8)

(c) Comment on the suitability of the model for large values of t.
(2)

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Question	Scheme	Marks	AOs
5(a)	$4m^2+4m+37=0 \Rightarrow m=-\frac{1}{2}\pm 3\text{i}$	M1	1.1b
	$h=\text{e}^{-0.5t}\left(A\cos 3t+B\sin 3t\right)$	A1	1.1b
		(2)	
(b)	$t=0,\ \ h=-20 \Rightarrow A=-20$	M1	3.4
	$\frac{\text{d}h}{\text{d}t}=-0.5\text{e}^{-0.5t}\left(A\cos 3t+B\sin 3t\right)+\text{e}^{-0.5t}\left(-3A\sin 3t+3B\cos 3t\right)$ $t=0,\ \frac{\text{d}h}{\text{d}t}=55 \Rightarrow B=\ldots$ (NB B = 15)	M1	3.4
	$(h=)\text{e}^{-0.5t}\left(15\sin 3t-20\cos 3t\right)$	A1	1.1b
	$-0.5\text{e}^{-0.5t}\left(15\sin 3t-20\cos 3t\right)+\text{e}^{-0.5t}\left(60\sin 3t+45\cos 3t\right)=0$ or e.g. $-0.5\text{e}^{-0.5t}\left(15\sin 3t-20\cos 3t\right)+\frac{25\sqrt{37}}{2}\text{e}^{-0.5t}\sin\left(3t+\arctan\frac{22}{21}\right)=0$ $\Rightarrow t=\ldots$	M1	3.1b
	$\tan 3t=-\frac{22}{21}$ or e.g. $3t+\tan^{-1}\frac{22}{21}=0$	A1 **M1 on ePEN**	2.1
	t = 0.778 s	A1	1.1b
	$h=\text{e}^{-0.5\times\text{"}0.778\text{"}}\left(15\sin\left(3\times\text{"}0.778\text{"}\right)-20\cos\left(3\times\text{"}0.778\text{"}\right)\right)$	**d**M1	1.1b
	= 16.7 cm	A1	3.2a
		(8)	
(c)	E.g. considers large values of t in the model for h **or** states that for large values of t, h becomes smaller or becomes zero	M1	3.4
	E.g. • The value of h is very small when t is large and this is likely to be correct (as the displacement of end of the board should get smaller and smaller) • This suggests the model is suitable • This is realistic • This is suitable as the board will tend towards its equilibrium position • When t is large the value of h is never zero so the model is not really appropriate for large values of t	A1 **B1 on ePEN**	3.2b
		(2)	
			(12 marks)

Notes

(a)

M1: Uses the model to form and solve the auxiliary equation $4m^2+4m+37=0$

See General Guidance for awarding this mark. This can be implied by correct values for m (from calculator)

A1: Correct general solution including "h ="

(b)

M1: Uses the model and the initial conditions to establish the value of "*A*". Need to see $t = 0$ and $h = \pm 20$ leading to a value for "*A*". This may be implied by $A = -20$ or $A = 20$. M1: Differentiates their model using the product rule and uses the initial conditions, $t = 0$ with $\mathrm{d}h/\mathrm{d}t = \pm 55$, to establish the value of "*B*" A1: Correct particular solution or correct values for *A* and *B* M1: Uses their solution to the model with a correct strategy to obtain a value for *t* e.g. differentiates or uses their derivative from earlier, sets equal to zero and solves for *t* A1(M1 on ePEN): Correct equation for *t* A1: Correct value for *t* (allow awrt 0.778 if necessary) but this value may be implied. **d**M1: Uses the model and their **positive** value for *t* to find the maximum displacement - **if their *t* is incorrect there must be some indication that they are using their *h* and not just a number written down. E.g. must see substitution into their *h* or they re-state their *h* and obtain a value for *h*.** **Dependent on all the previous method marks** A1: Correct value (awrt 16.7 (units not needed)) (c) M1: Considers the model for large values of *t* either by substituting values or by considering the expression and commenting on its behaviour for large values of *t*. E.g. as $t \to \infty,\ h \to 0$ or as $t \to \infty,\ e^{-0.5t} \to 0$ or as $t \to \infty$ the oscillations become smaller etc. A1: Makes a suitable comment – see scheme for examples

7. A sample of bacteria in a sealed container is being studied.

The number of bacteria, P, in thousands, is modelled by the differential equation

$$(1+t)\frac{dP}{dt} + P = t^{\frac{1}{2}}(1+t)$$

where t is the time in hours after the start of the study.

Initially, there are exactly 5000 bacteria in the container.

(a) Determine, according to the model, the number of bacteria in the container 8 hours after the start of the study.

(6)

(b) Find, according to the model, the rate of change of the number of bacteria in the container 4 hours after the start of the study.

(4)

(c) State a limitation of the model.

(1)

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P 6 2 6 7 2 A 0 2 4 2 8

Question	Scheme	Marks	AOs
7(a)	$(1+t)\frac{dP}{dt}+P=t^{\frac{1}{2}}(1+t)\Rightarrow\frac{dP}{dt}+\frac{P}{1+t}=t^{\frac{1}{2}}$	B1	1.1b
	$I=e^{\int\frac{1}{1+t}dt}=1+t\Rightarrow P(1+t)=\int t^{\frac{1}{2}}(1+t)dt=...$	M1	3.1b
	$P(1+t)=\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+c$	A1	1.1b
	$t=0,\ P=5\Rightarrow c=5$	M1	3.4
	$P=\frac{\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+5}{(1+t)}=\frac{\frac{2}{3}8^{\frac{3}{2}}+\frac{2}{5}8^{\frac{5}{2}}+5}{9}=...$	M1	1.1b
	= 10 277 bacteria (allow awrt 10 300)	A1	2.2b
		(6)	
(b)	$P=\frac{\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+5}{(1+t)}\Rightarrow\frac{dP}{dt}=\frac{(1+t)\left(t^{\frac{1}{2}}+t^{\frac{3}{2}}\right)-\left(\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+5\right)}{(1+t)^2}$ Alt: $P+(1+t)\frac{dP}{dt}=t^{\frac{1}{2}}+t^{\frac{3}{2}}\Rightarrow\frac{dP}{dt}=\frac{t^{\frac{1}{2}}+t^{\frac{3}{2}}-\frac{\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+5}{(1+t)}}{(1+t)}$	M1 A1ft	3.4 1.1b
	$\left(\frac{dP}{dt}\right)_{t=1}=\frac{dP}{dt}=\frac{5\times10-\left(\frac{16}{3}+\frac{64}{5}+5\right)}{(5)^2}=\frac{403}{375}$	dM1	3.1a
	$\frac{403}{375}\times1000=\frac{3224}{3}(=awrt\,1070)$ bacteria per hour	A1	3.2a
		(4)	
	(b) Alternative:		
	$P=\frac{\frac{2}{3}t^{\frac{3}{2}}+\frac{2}{5}t^{\frac{5}{2}}+5}{(1+t)}=\frac{\frac{16}{3}+\frac{64}{5}+5}{(1+4)}$	M1	3.4
	$=\frac{347}{75}$	A1ft	1.1b
	$(1+t)\frac{dP}{dt}+P=t^{\frac{1}{2}}(1+t)\Rightarrow5\frac{dP}{dt}+\frac{347}{75}=2\times5\Rightarrow\frac{dP}{dt}=\frac{403}{375}$	dM1	3.1a
	$\frac{403}{375}\times1000=\frac{3224}{3}(=1075)$ bacteria per hour	A1	3.2a
		(4)	
(c)	E.g. • The number of bacteria increases indefinitely which is not realistic	B1	3.5b
		(1)	
			(11 marks)

Notes
(a) B1: A correct rearrangement (may be implied by subsequent work). Alternatively, recognises the LHS as a derivative and writes $(1+t)\frac{\mathrm{d}P}{\mathrm{d}t}+P=\frac{\mathrm{d}}{\mathrm{d}t}\left(P(1+t)\right)\left(=t^{\frac{1}{2}}(1+t)\right)$ (may be implied). M1: Uses the model to find the integrating factor (or recognise the derivative) and attempts the solution of the differential equation to achieve $P\times \text{their IF}=\int \text{their } t^{\frac{1}{2}}\times \text{their IF d}t=...$ but do not be too concerned with the mechanics of integrating the RHS but it must be attempted. A1: Correct solution M1: Interprets the initial conditions to find the constant of integration. Must be using t = 0 and P = 5 in an equation with a constant of integration, but their equation may have come from incorrect work. This is correctly interpreting the initial conditions and attempting to use them. M1: Uses their solution to the problem to find the population after 8 hours. Must be using their solution, but allow for any equations which arise from an attempt at solving the differential equation. A1: **cso** Correct number of bacteria (accept awrt 10 300) from a correct equation (b) M1: Realises the need to differentiate the model and uses an appropriate method to find the derivative. Allow the M for attempts at implicit differentiation with (1+t)P = ... Trivialised differentiation from incorrect work is M0. A1ft: Correct differentiation of the correct answer to (a) up to the constant of integration to obtain dP/dt in terms of t (if implicit differentiation is used, they must get to a function in terms of t only, or revert to the Alternative method). Follow through on their c in an otherwise correct equation from (a). M1: Uses t = 4 in their dP/dt (allow from any attempts at the derivative) to obtain a value for dP/dt. A1: Correct answer, allow 1075 or answers rounding down to 1070 with correct units. Accept as 1.07 thousand bacteria per hour. (NB If 5000 is used in (a) instead of 5, the answer here would be −198.725) **Alternative:** M1: Substitutes t = 4 into their P A1ft: Correct value for P. Follow through on their constant of integration from part (a), but the rest of the equation must be correct. M1: Uses t = 4 and their P to find a value for dP/dt A1: Correct answer allow 1075 or answers rounding down to 1070 with correct units. Accept as 1.07 thousand bacteria per hour. (c) B1: Suggests a suitable limitation which must refer to the model. Allow for a sensible comment even if they have no equation for the model Do not allow answers such as "the model does not take account of external factors such as temperature" as we do not know what factors the model does take account of.

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3. (a) Determine the general solution of the differential equation

$$\cos x \frac{dy}{dx} + y \sin x = e^{2x} \cos^2 x$$

giving your answer in the form $y = f(x)$

(3)

Given that $y = 3$ when $x = 0$

(b) determine the smallest positive value of x for which $y = 0$

(3)

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Question	Scheme	Marks	AOs
3(a)	$\dfrac{dy}{dx} + y\tan x = e^{2x}\cos x$ IF $= e^{\int \tan x\, dx} = e^{\ln\sec x} = \sec x \Rightarrow \sec x\dfrac{dy}{dx} + y\sec x\tan x = e^{2x}$ $\Rightarrow y\sec x = \int e^{2x}\ dx$	M1	3.1a
	$y\sec x = \dfrac{1}{2}e^{2x}(+c)$	A1	1.1b
	$y = \left(\dfrac{1}{2}e^{2x} + c\right)\cos x$	A1	1.1b
		(3)	
(b)	$x = 0,\ y = 3 \Rightarrow c = \ldots\{2.5\}$	M1	3.1a
	$y = \left(\dfrac{1}{2}e^{2x} + \dfrac{5}{2}\right)\cos x = 0 \Rightarrow \cos x = 0 \Rightarrow x = \ldots$	M1	1.1b
	$x = \dfrac{\pi}{2}$	A1	1.1b
		(3)	
			(6 marks)

Notes:

(a)

M1: Finds the integrating factor and attempts the solution of the differential equation.

Look for I.F. $= e^{\int \tan x dx} \Rightarrow y \times$ 'their I.F.' $= \int e^{2x}\cos x \times$ 'their I.F.' dx

A1: Correct solution condone missing + *c*

A1: Correct general solution, Accept equivalents of the form *y* = f(*x*), such as $y = \dfrac{e^{2x}}{2\sec x} + \dfrac{c}{\sec x}$

(b)

M1: Uses $x = 0\quad y = 3$ to find the constant of integration. Allow if done as part of part (a) and allow for their answer to (a) as long as it has a constant of integration to find.

M1: Sets $y = 0$ in an equation of the form $y = (Ae^{2x} + c)\cos x$ (oe) where *A* is 1, 2 or $\dfrac{1}{2}$, with their *c* or constant *c* and makes a valid attempt to solve the equation to find a value for *x*. (Allow even if the constant of integration has not been found).

A1: Depends on both M's. Awrt 1.57 or $\frac{\pi}{2}$only. There must have been an attempt to find the constant of integration, but allow from a correct answer to (a) as long as a positive value for *c* has been found (can be scored from implicit form).