Cp2Ch5 XMQs and MS

(Total: 75 marks)

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1. CP1_Sample   Q4 .  9 marks - CP2ch5 Polar coordinates

2. CP2_Sample   Q6 . 13 marks - CP2ch5 Polar coordinates

3. CP1_Specimen Q3 . 10 marks - CP2ch5 Polar coordinates

4. CP1_2019     Q3 . 10 marks - CP2ch5 Polar coordinates

5. CP1_2020     Q3 .  9 marks - CP2ch5 Polar coordinates

6. CP2_2021     Q6 . 14 marks - CP2ch5 Polar coordinates

7. CP2_2022     Q7 . 10 marks - CP2ch5 Polar coordinates
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4.

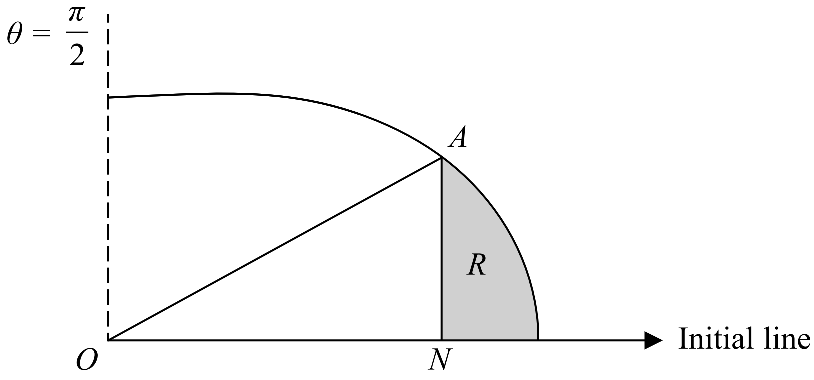


Figure 1

The curve C shown in Figure 1 has polar equation

$$r = 4 + \cos 2\theta \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

At the point A on C, the value of r is $\frac{9}{2}$

The point N lies on the initial line and AN is perpendicular to the initial line.

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the initial line and the line AN.

Find the exact area of the shaded region R, giving your answer in the form $p\pi + q\sqrt{3}$ where p and q are rational numbers to be found.

(9)

Question	Scheme	Marks	AOs
4	$4+\cos 2\theta=\frac{9}{2}\Rightarrow\theta=...$	M1	3.1a
	$\theta=\frac{\pi}{6}$	A1	1.1b
	$\frac{1}{2}\int(4+\cos 2\theta)^2\,\mathrm{d}\theta=\frac{1}{2}\int\left(16+8\cos 2\theta+\cos^2 2\theta\right)\mathrm{d}\theta$	M1	3.1a
	$\cos^2 2\theta=\frac{1}{2}+\frac{1}{2}\cos 4\theta\Rightarrow A=\frac{1}{2}\int\left(16+8\cos 2\theta+\frac{1}{2}+\frac{1}{2}\cos 4\theta\right)\mathrm{d}\theta$	M1	3.1a
	$=\frac{1}{2}\left[16\theta+4\sin 2\theta+\frac{\sin 4\theta}{8}+\frac{\theta}{2}\right]$	A1	1.1b
	Using limits 0 and their $\frac{\pi}{6}$: $\frac{1}{2}\left[\frac{33\pi}{12}+2\sqrt{3}+\frac{\sqrt{3}}{16}-(0)\right]$	M1	1.1b
	Area of triangle = $\frac{1}{2}(r\cos\theta)(r\sin\theta)=\frac{1}{2}\times\frac{81}{4}\times\frac{1}{2}\times\frac{\sqrt{3}}{2}$	M1	3.1a
	Area of $R=\frac{33\pi}{24}+\frac{33\sqrt{3}}{32}-\frac{81\sqrt{3}}{32}$	M1	1.1b
	$=\frac{11}{8}\pi-\frac{3\sqrt{3}}{2}\left(p=\frac{11}{8},\ q=-\frac{3}{2}\right)$	A1	1.1b
			(9 marks)

Notes:

M1: Realises the angle for A is required and attempts to find it
A1: Correct angle
M1: Uses a correct area formula and squares r to achieve a 3TQ integrand in $\cos 2\theta$
M1: Use of the correct double angle identity on the integrand to achieve a suitable form for integration
A1: Correct integration
M1: Correct use of limits
M1: Identifies the need to subtract the area of a triangle and so finds the area of the triangle
M1: Complete method for the area of R
A1: Correct final answer

6. (a) (i) Show on an Argand diagram the locus of points given by the values of z satisfying

$$|z - 4 - 3\mathbf{i}| = 5$$

Taking the initial line as the positive real axis with the pole at the origin and given that $\theta \in [\alpha, \alpha + \pi]$, where $\alpha = -\arctan\left(\frac{4}{3}\right)$,

(ii) show that this locus of points can be represented by the polar curve with equation

$$r = 8\cos\theta + 6\sin\theta$$

(6)

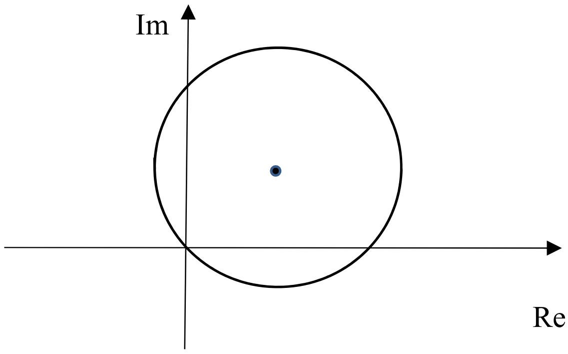
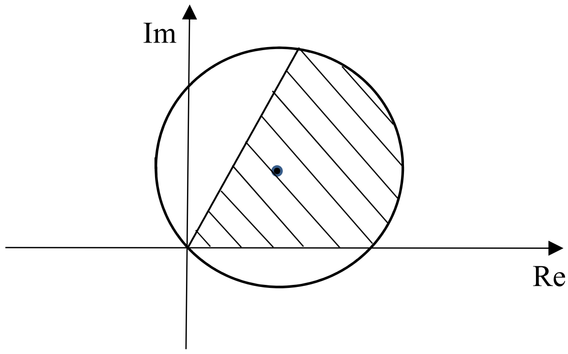
The set of points A is defined by

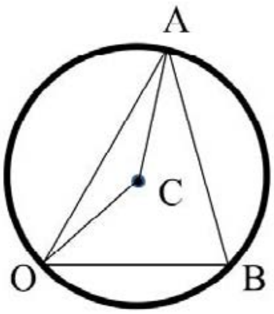
$$A = \left\{z : 0 \leqslant \arg z \leqslant \frac{\pi}{3}\right\} \cap \left\{z : |z - 4 - 3\mathbf{i}| \leqslant 5\right\}$$

(b) (i) Show, by shading on your Argand diagram, the set of points A.

(ii) Find the **exact** area of the region defined by A, giving your answer in simplest form.

(7)

Question	Scheme	Marks	AOs
6(a)(i)		M1	1.1b
		A1	1.1b
(a)(ii)	$\lvert z-4-3\mathrm{i}\rvert=5\Rightarrow\lvert x+\mathrm{i}y-4-3\mathrm{i}\rvert=5\Rightarrow(x-4)^2+(y-3)^2=...$	M1	2.1
	$(x-4)^2+(y-3)^2=25$ or any correct form	A1	1.1b
	$(r\cos\theta-4)^2+(r\sin\theta-3)^2=25$ $\Rightarrow r^2\cos^2\theta-8r\cos\theta+16+r^2\sin^2\theta-6r\sin\theta+9=25$ $\Rightarrow r^2-8r\cos\theta-6r\sin\theta=0$	M1	2.1
	$\therefore r=8\cos\theta+6\sin\theta$ *	A1*	2.2a
		(6)	
(b)(i)		B1	1.1b
		B1ft	1.1b
(b)(ii)	$A=\frac{1}{2}\int r^2\,\mathrm{d}\theta=\frac{1}{2}\int(8\cos\theta+6\sin\theta)^2\,\mathrm{d}\theta$ $=\frac{1}{2}\int(64\cos^2\theta+96\sin\theta\cos\theta+36\sin^2\theta)\mathrm{d}\theta$	M1	3.1a
	$=\frac{1}{2}\int(32(\cos2\theta+1)+96\sin\theta\cos\theta+18(1-\cos2\theta))\mathrm{d}\theta$	M1	1.1b
	$=\frac{1}{2}\int(14\cos2\theta+50+48\sin2\theta)\mathrm{d}\theta$	A1	1.1b
	$=\frac{1}{2}[7\sin2\theta+50\theta-24\cos2\theta]_0^{\frac{\pi}{3}}=\frac{1}{2}\left\{\left(\frac{7\sqrt{3}}{2}+\frac{50\pi}{3}+12\right)-(-24)\right\}$	M1	2.1
	$=\frac{7\sqrt{3}}{4}+\frac{25\pi}{3}+18$	A1	1.1b
		(7)	

Question	Scheme	Marks	AOs
	(b)(ii) Alternative:  Candidates may take a geometric approach e.g. by finding sector + 2 triangles		
	Angle $ACB = \left(\frac{2\pi}{3}\right)$ so area sector $ACB = \frac{1}{2}(5)^2\frac{2\pi}{3}$ Area of triangle $OCB = \frac{1}{2}\times8\times3$	M1	3.1a
	Sector area ACB + triangle area $OCB = \frac{25\pi}{3} + 12$	A1	1.1b
	Area of triangle OAC: Angle $ACO = 2\pi - \frac{2\pi}{3} - \cos^{-1}\left(\frac{5^2+5^2-8^2}{2\times5\times5}\right)$ so area $OAC = \frac{1}{2}(5)^2\sin\left(\frac{4\pi}{3} - \cos^{-1}\left(\frac{-7}{25}\right)\right)$	M1	1.1b
	$= \frac{25}{2}\left(\sin\frac{4\pi}{3}\cos\left(\cos^{-1}\left(\frac{-7}{25}\right)\right) - \cos\frac{4\pi}{3}\sin\left(\cos^{-1}\left(\frac{-7}{25}\right)\right)\right)$ $= \frac{25}{2}\left(\left(\frac{7\sqrt{3}}{50}\right) + \frac{1}{2}\sqrt{1-\left(\frac{7}{25}\right)^2}\right) = \frac{7\sqrt{3}}{4} + 6$ Total area $= \frac{25\pi}{3} + \frac{1}{2}\times8\times3 + 6 + \frac{7\sqrt{3}}{4}$	M1	2.1
	$= \frac{7\sqrt{3}}{4} + \frac{25\pi}{3} + 18$	A1	1.1b
			(13 marks)

Question 6 notes:	
(a)(i)	
M1:	Draws a circle which passes through the origin
A1:	Fully correct diagram
(a)(ii)	
M1:	Uses $z = x + \mathrm{i}y$ in the given equation and uses modulus to find equation in x and y only
A1:	Correct equation in terms of x and y in any form – may be in terms of r and θ
M1:	Introduces polar form, expands and uses $\cos^2\theta + \sin^2\theta = 1$ leading to a polar equation
A1*:	Deduces the given equation (ignore any reference to $r = 0$ which gives a point on the curve)
(b)(i)	
B1:	Correct pair of rays added to their diagram
B1ft:	Area between their pair of rays and inside their circle from (a) shaded, as long as there is an intersection
(b)(ii)	
M1:	Selects an appropriate method by linking the diagram to the polar curve in (a), evidenced by use of the polar area formula
M1:	Uses double angle identities
A1:	Correct integral
M1:	Integrates and applies limits
A1:	Correct area
(b)(ii) Alternative:	
M1:	Selects an appropriate method by finding angle ACB and area of sector ACB and finds area of triangle OCB to make progress towards finding the required area
A1:	Correct combined area of sector ACB + triangle OCB
M1:	Starts the process of finding the area of triangle OAC by calculating angle ACO and attempts area of triangle OAC
M1:	Uses the addition formula to find the exact area of triangle OAC and employs a full correct method to find the area of the shaded region
A1:	Correct area

3.

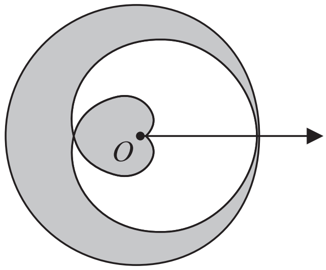


Figure 1

Figure 1 shows a sketch for the design of a logo. The logo is defined by the polar curve with equation

$$r = \sin\left(\frac{\theta}{6}\right) \qquad 0 \leqslant \theta \leqslant 6\pi$$

The inner closed section and outer closed section of the curve, shown shaded in Figure 1, are to be coloured the same colour. The remaining section is to be left clear.

(a) Use algebraic integration to find the area of the coloured sections of the logo.

(6)

A copy of this logo is to be painted on a white wall of a building such that the total width of the logo is 12 m.

Tins of coloured paint with an advertised minimum coverage area of 30 m^2 are to be used to paint the coloured sections of the logo onto the wall. Given that two coats of paint will be needed,

(b) find the minimum number of tins of this paint that should be bought to ensure that the coloured sections of the logo can be painted onto the wall.

(4)

Question	Scheme	Marks	AOs
3(a)	Correct overall strategy employed, eg. $A=2\times\left(\frac{1}{2}\int_{2\pi}^{3\pi}\sin^2\left(\frac{\theta}{6}\right)d\theta-\frac{1}{2}\int_{4\pi}^{5\pi}\sin^2\left(\frac{\theta}{6}\right)d\theta+\frac{1}{2}\int_{0}^{\pi}\sin^2\left(\frac{\theta}{6}\right)d\theta\right)$	M1	3.1a
	Evidence of use of $\frac{1}{2}\int\sin^2\left(\frac{\theta}{6}\right)d\theta$	B1	1.1a
	$\int\sin^2\left(\frac{\theta}{6}\right)d\theta=\int\frac{1}{2}\left(1-\cos\left(\frac{\theta}{3}\right)\right)d\theta$	M1	3.1a
	$=\frac{1}{2}\left(\theta-3\sin\left(\frac{\theta}{3}\right)\right)$	A1	1.1b
	$A=\left(2\times\frac{1}{2}\right)\times\frac{1}{2}\left[\left((3\pi-0)-\left(2\pi-\frac{3\sqrt{3}}{2}\right)\right)-\left(\left(5\pi+\frac{3\sqrt{3}}{2}\right)-\left(4\pi+\frac{3\sqrt{3}}{2}\right)\right)+\left(\left(\pi-\frac{3\sqrt{3}}{2}\right)-(0)\right)\right]$	M1	2.1
	$=\frac{\pi}{2}$	A1	1.1b
		(6)	
(b)	Area of painting on wall = (area curve)×(12/(width of curve))^2 with their area and width.	M1	3.1a
	Width of curve $\left(=\sin\left(\frac{3\pi}{6}\right)+\sin\left(\frac{2\pi}{6}\right)\right)=1+\frac{\sqrt{3}}{2}=1.866...$	B1	1.1b
	So as two coats needed, total area of paint required $=2\times\frac{\pi}{2}\times41.354$ $=129.92\ldots\ \text{m}^2$	M1	2.2a
	So 5 tins of paint will be needed.	A1	3.2a
		(4)	
			(10 marks)

Notes:

(a)

M1: A correct attempt to find the correct area, splitting into suitable required sections and attempting the integration, combining correctly. There are many variations that could be used so check carefully the strategy is correct.

B1: Applies the area formula to the curve with any limits. The ½ may be implied if a correct overall formula is used.

M1: Applies the double angle formula to set up the integral

A1: Correct integration of $\sin^2(\theta/6)$

M1: Applies limits correctly to all of their integrals and combines and simplifies.

A1: Correct area from a fully correct method.

(b)

M1: Full attempt to find the appropriate scaling factor (must be squaring) to scale the area found in (a) to the area required for the wall painting.

B1: Correct width of logo in the curve found.

M1: Area of paint required for two coats found.

A1: Correct number of tins identified. Must be an integer answer.

3.

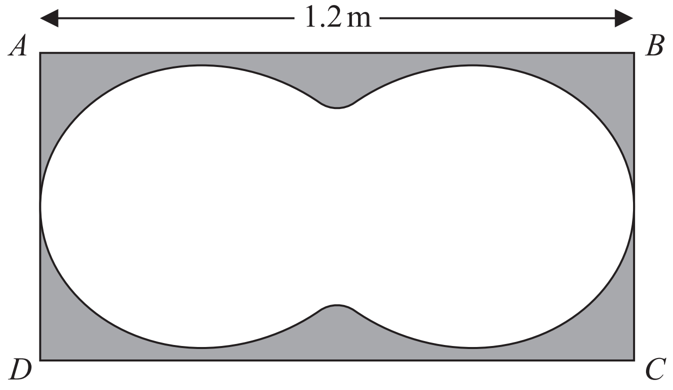


Diagram not to scale

Figure 1

Figure 1 shows the design for a table top in the shape of a rectangle $ABCD$. The length of the table, AB, is 1.2 m. The area inside the closed curve is made of glass and the surrounding area, shown shaded in Figure 1, is made of wood.

The perimeter of the glass is modelled by the curve with polar equation

$$r = 0.4 + a\cos 2\theta \qquad 0 \leqslant \theta < 2\pi$$

where a is a constant.

(a) Show that $a = 0.2$

(2)

Hence, given that $AD = 60$ cm,

(b) find the area of the wooden part of the table top, giving your answer in m^2 to 3 significant figures.

(8)

A1ft: Integrates $\int \frac{px}{2x^2+3} - \frac{q}{x+1}\,dx = \frac{p}{4}\ln\left(2x^2+3\right) - q\ln\left(x+1\right)$ and no extra terms

M1: Combines two algebraic log terms correctly.
B1: Correct upper limit for $x \to \infty$ by recognising the dominant terms. (Simply replacing x with ∞ scores B0). This can be implied.

A1: Deduces the correct value for the improper integral in the correct form, cao A0 for $2\ln\frac{2}{3}$

Correct answer with no working seen is no marks.
Note: Incorrect partial fraction form,
$\frac{A}{2x^2+3} + \frac{B}{x+1}$ or $\frac{Ax}{2x^2+3} + \frac{B}{x+1}$ the maximum it can score is M0M0A0A0M1B1A0

Question	Scheme	Marks	AOs
3(a)(i)	$2\left(0.4+a\right)=1.2$ or $0.4+a=0.6$ or $0.4+a\cos 0=0.6$ $\Rightarrow a = ...$	M1	3.4
	$a = 0.2$ * cso	A1*	1.1b
		(2)	
(b)	Area of rectangle is $1.2\times 0.6\left(=0.72\right)$	B1	1.1b
	Area enclosed by curve = $\frac{1}{2}\int \left(0.4+0.2\cos 2\theta\right)^2 \left(d\theta\right)$	M1	3.1a
	$\left(0.4+0.2\cos 2\theta\right)^2 = 0.16+0.16\cos 2\theta+0.04\cos^2 2\theta$ $= 0.16+0.16\cos 2\theta+0.04\left(\frac{\cos 4\theta+1}{2}\right)$	M1	2.1
	$\frac{1}{2}\int \left(0.4+0.2\cos 2\theta\right)^2 d\theta = \frac{1}{2}\left[0.18\theta+0.08\sin 2\theta+0.005\sin 4\theta\left(+c\right)\right]$ $=0.09\theta+0.04\sin 2\theta+0.0025\sin 4\theta\left(+c\right)$ o.e.	A1ft	1.1b
	Area enclosed by curve = $\left[0.09\theta+0.04\sin 2\theta+0.0025\sin 4\theta\right]_0^{2\pi}$ or Area enclosed by curve = $2\left[0.09\theta+0.04\sin 2\theta+0.0025\sin 4\theta\right]_0^{\pi}$ or Area enclosed by curve = $4\left[0.09\theta+0.04\sin 2\theta+0.0025\sin 4\theta\right]_0^{\pi/2}$	dM1	3.1a
	$=\frac{9}{50}\pi$ or $0.18\pi\left(=0.5654...\right)$	A1	1.1b

	Area of wood $=1.2\times0.6-0.18\pi$	M1	1.1b
	$=$ awrt 0.155 (m^2)	A1	1.1b
		(8)	
			(10 marks)

Notes

(a)
M1: Interprets the information from the model and realises that the maximum value of *r* gives half the length of the table top (or equivalent) and solves to find a value for *a*. Use $\theta=0$ and $r=0.6$ or $\theta=\pi$ and $r=-0.6$ to find a value for *a*.
Using $\theta=2\pi$ is M0
A1*: Correct value for *a*.

Alternative

M1: Uses $a = 0.2$ and $\theta=0$ to find a value for *r*
A1: Finds $r = 0.6$ and concludes that $a = 0.2$

(b)
B1: 1.2×0.6 or 0.72
M1: A correct strategy identified for finding an area enclosed by the polar curve using a correct formula with *r* substituted. Attempt at area $=\frac{1}{2}\int (0.4+0.2\cos 2\theta)^2\,\text{d}\theta=...$

Look for $=\lambda\times\frac{1}{2}\int (0.4+0.2\cos 2\theta)^2\,\text{d}\theta=...$

If the $\frac{1}{2}$ is not explicitly seen then look at the limits and it must be either

$$=\int_0^{\pi}(0.4+0.2\cos 2\theta)^2\,\text{d}\theta=... \text{ or } =2\int_0^{\frac{\pi}{2}}(0.4+0.2\cos 2\theta)^2\,\text{d}\theta=...$$

Condone missing $\text{d}\theta$

M1: Squares to achieve three terms and uses $\cos^2 2\theta=\frac{\pm1\pm\cos 4\theta}{2}$ to obtain an expression in an integrable form.
A1ft: Correct follow through integration as long as the previous two method marks have been awarded.
dM1: Dependent of first method mark. Finds the required area enclosed by the curve using the correct limits.

There are only three cases either $\frac{1}{2}\int_0^{2\pi}(0.4+0.2\cos 2\theta)^2\,\text{d}\theta$ or $\int_0^{\pi}(0.4+0.2\cos 2\theta)^2\,\text{d}\theta$ or

$2\int_0^{\frac{\pi}{2}}(0.4+0.2\cos 2\theta)^2\,\text{d}\theta$

The use of the limit 0 can be implied if it gives 0 but the use of 0 must been seen or implied if it does not result in 0 (just writing 0 is insufficient)

A1: Correct area of the glass following fully correct working. **Do not award for the correct answer following incorrect working.**
M1: Subtracts their area of the glass from their area of the rectangle, as long as it does not give a negative area
A1: awrt 0.155 or awrt $0.155\,\text{m}^2$ (If the units are stated they must be correct)

Note: Using a calculator to find the area scores a maximum of B1M0M0A0M0A0M1A1

Question	Scheme	Marks	AOs
4	$\frac{1}{(r+1)(r+2)(r+3)} \equiv \frac{A}{r+1}+\frac{B}{r+2}+\frac{C}{r+3} \Rightarrow A=..., B=..., C=...$ $\left(\text{NB } A=\frac{1}{2} \quad B=-1 \quad C=\frac{1}{2}\right)$	M1	3.1a
	$r=0$ $\quad \frac{1}{2}\left[\frac{1}{1}-\frac{2}{2}+\frac{1}{3}\right]$ or $\frac{1}{2\ 1}-\frac{1}{2}+\frac{1}{2\ 3}$ or $\frac{1}{2}-\frac{1}{2}+\frac{1}{6}$ $r=1$ $\quad \frac{1}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]$ or $\frac{1}{2\ 2}-\frac{1}{3}+\frac{1}{2\ 4}$ or $\frac{1}{4}-\frac{1}{3}+\frac{1}{8}$ $r=n-1$ $\quad \frac{1}{2}\left[\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right]$ or $\frac{1}{2\ n}-\frac{1}{n+1}+\frac{1}{2\ n+2}$ or $\frac{1}{2n}-\frac{1}{n+1}+\frac{1}{2n+4}$ $r=n$ $\quad \frac{1}{2}\left[\frac{1}{n+1}-\frac{2}{n+2}+\frac{1}{n+3}\right]$ or $\frac{1}{2\ n+1}-\frac{1}{n+2}+\frac{1}{2\ n+3}$ or $\frac{1}{2n+2}-\frac{1}{n+2}+\frac{1}{2n+6}$	M1	2.1
	$\frac{1}{2}-\frac{1}{2}+\frac{1}{4}+\frac{1}{2(n+2)}-\frac{1}{n+2}+\frac{1}{2(n+3)}$ or $\frac{1}{4}-\frac{1}{2(n+2)}+\frac{1}{2(n+3)}$	A1	1.1b
	$=\frac{n^2+5n+6+2n+6-4n-12+2n+4}{4(n+2)(n+3)}$	M1	1.1b
	$=\frac{(n+1)(n+4)}{4(n+2)(n+3)}$	A1	2.2a
		(5)	
			(5 marks)

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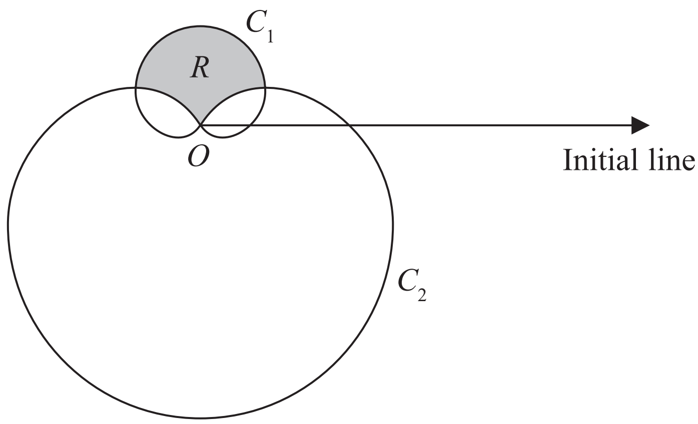


Figure 1

Figure 1 shows a sketch of two curves C_1 and C_2 with polar equations

$$C_1: r = (1 + \sin\theta) \qquad 0 \leqslant \theta < 2\pi$$

$$C_2: r = 3(1 - \sin\theta) \qquad 0 \leqslant \theta < 2\pi$$

The region R lies inside C_1 and outside C_2 and is shown shaded in Figure 1.

Show that the area of R is

$$p\sqrt{3} - q\pi$$

where p and q are integers to be determined.

(9)

Question	Scheme	Marks	AOs
3	$3(1-\sin\theta)=1+\sin\theta \Rightarrow \sin\theta=\frac{1}{2} \Rightarrow \theta=\ldots$	M1	3.1a
	$\theta=\frac{\pi}{6}\left(\text{or } \frac{5\pi}{6}\right)$	A1	1.1b
	Use of $\frac{1}{2}\int(1+\sin\theta)^2\,\mathrm{d}\theta$ or $\frac{1}{2}\int\{3(1-\sin\theta)\}^2\,\mathrm{d}\theta$	M1	1.1a
	$\left(\frac{1}{2}\right)\int\left[(1+\sin\theta)^2-9(1-\sin\theta)^2\right]\mathrm{d}\theta$ $=\left(\frac{1}{2}\right)\int\left[1+2\sin\theta+\sin^2\theta-9+18\sin\theta-9\sin^2\theta\right]\mathrm{d}\theta$ or $\int(1+\sin\theta)^2\,\mathrm{d}\theta=\int\left(1+2\sin\theta+\sin^2\theta\right)\mathrm{d}\theta$ **and** $\int 9(1-\sin\theta)^2\,\mathrm{d}\theta=9\int\left(1-2\sin\theta+\sin^2\theta\right)\mathrm{d}\theta$	M1 A1	2.1 1.1b
	$\int\sin^2\theta\,\mathrm{d}\theta=\frac{1}{2}\int(1-\cos 2\theta)\,\mathrm{d}\theta \Rightarrow$ $\int\left[(1+\sin\theta)^2-9(1-\sin\theta)^2\right]\mathrm{d}\theta=2\sin 2\theta-12\theta-20\cos\theta$	M1 A1	3.1a 1.1b
	$A=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\left[(1+\sin\theta)^2-9(1-\sin\theta)^2\right]\mathrm{d}\theta$ or $A=2\times\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left[(1+\sin\theta)^2-9(1-\sin\theta)^2\right]\mathrm{d}\theta$ $=\frac{1}{2}\left\{\left(-\sqrt{3}-10\pi+10\sqrt{3}\right)-\left(\sqrt{3}-2\pi-10\sqrt{3}\right)\right\}=\ldots$	DM1	3.1a
	$=9\sqrt{3}-4\pi$	A1	1.1b
		(9)	
			(9 marks)

Notes

M1: Realises that the angles at the intersection are required and solves $C_1 = C_2$ to obtain a value for θ

A1: Correct value for θ. Must be in radians – if given in degrees you may need to check later to see if they convert to radians before substitution.

M1: Evidence selecting the correct polar area formula on either curve

M1: Fully expands both expressions for r^2 either as parts of separate integrals or as one complete integral. (Can be scored from incorrect polar area formula, e.g. missing the ½)

A1: Correct expansions for both curves (may be unsimplified)

M1: Selects the correct strategy by applying the correct double angle identity in order to reach an integrable form and attempting the integration of at least one of the curves.
A1: Correct integration (of both integrals if done separately),
FYI: If done separately the correct integrals are

$$\int (1+\sin\theta)^2 \, d\theta = \theta - 2\cos\theta + \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) = \frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta \text{ and}$$

$$\int 9(1-\sin\theta)^2 \, d\theta = 9\theta + 18\cos\theta + \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) = \frac{27}{2}\theta + 18\cos\theta - \frac{9}{4}\sin 2\theta$$

DM1: Depends on all previous M's. For a fully correct strategy with appropriate limits correctly applied to their integral or integrals and terms combined if necessary. Make sure that if limits of $\frac{\pi}{6}$ and $\frac{\pi}{2}$ are used that the area is doubled as part of the strategy.
A1: Correct area

6. The curve C has equation

$$r = a(p + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi$$

where a and p are positive constants and $p > 2$

There are exactly four points on C where the tangent is perpendicular to the initial line.

(a) Show that the range of possible values for p is

$$2 < p < 4$$

(5)

(b) Sketch the curve with equation

$$r = a(3 + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi \qquad \text{where } a > 0$$

(1)

John digs a hole in his garden in order to make a pond.

The pond has a uniform horizontal cross section that is modelled by the curve with equation

$$r = 20(3 + 2\cos\theta) \qquad 0 \leqslant \theta < 2\pi$$

where r is measured in centimetres.

The depth of the pond is 90 centimetres.

Water flows through a hosepipe into the pond at a rate of 50 litres per minute.

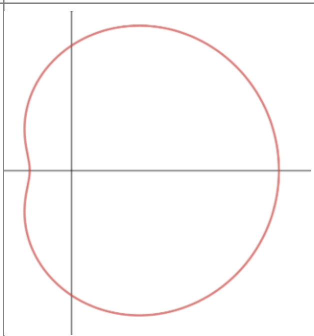
Given that the pond is initially empty,

(c) determine how long it will take to completely fill the pond with water using the hosepipe, according to the model. Give your answer to the nearest minute.

(7)

(d) State a limitation of the model.

(1)

Question	Scheme	Marks	AOs
6(a)	$x = r\cos\theta = a\left(p + 2\cos\theta\right)\cos\theta$ Leading to $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = \alpha\sin\theta\cos\theta + \beta\sin\theta\left(p + 2\cos\theta\right)$ or $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = \alpha\sin\theta\cos\theta + \beta\sin\theta$ or $x = a\left(p\cos\theta + 2\cos^2\theta\right) = a\left(\cos 2\theta + p\cos\theta + 1\right)$ leading to $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = \alpha\sin 2\theta + \beta\sin\theta$	M1	3.1a
	$\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = a\left[-2\sin\theta\cos\theta - \sin\theta\left(p + 2\cos\theta\right)\right]$ or $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -4a\sin\theta\cos\theta - ap\sin\theta$ or $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -2a\sin 2\theta - ap\sin\theta$	A1	1.1b
	$a\left[-2\sin\theta\cos\theta - \sin\theta\left(p + 2\cos\theta\right)\right] = 0$ $\pm a\left(4\sin\theta\cos\theta + p\sin\theta\right) = 0$ $a\sin\theta\left(4\cos\theta + p\right) = 0$ Either $\sin\theta = 0$ or $\cos\theta = -\dfrac{p}{4}$	M1	3.1a
	$\sin\theta = 0$ implies 2 solutions (tangents which are perpendicular to the initial line) e.g. $\theta = 0, \pi$	B1	2.2a
	Therefore two solutions to $\cos\theta = -\dfrac{p}{4}$ are required $-\dfrac{p}{4} > -1 \Rightarrow p < 4$ as p is a positive constant $2 < p < 4$*	A1*	2.4
		(5)	
(b)	 Correct shape and position. Condone cusp	B1	2.2a
		(1)	
(c)	Area = $2 \times \dfrac{1}{2}\int_0^{\pi}\left[20\left(3 + 2\cos\theta\right)\right]^2\,\mathrm{d}\theta = 400\int_0^{\pi}\left(9 + 12\cos\theta + 4\cos^2\theta\right)\mathrm{d}\theta$ or $= \int_0^{\pi}\left(3600 + 4800\cos\theta + 1600\cos^2\theta\right)\mathrm{d}\theta$	M1	3.4

	or $$\frac{1}{2}\int_0^{2\pi}\left[20\left(3+2\cos\theta\right)\right]^2\,\mathrm{d}\theta=200\int_0^{2\pi}\left(9+12\cos\theta+4\cos^2\theta\right)\mathrm{d}\theta$$ $$\text{or }=\int_0^{2\pi}\left(1800+2400\cos\theta+800\cos^2\theta\right)\mathrm{d}\theta$$		
	$\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta\Rightarrow$ $A=...\int\left(9+12\cos\theta+2+2\cos 2\theta\right)\mathrm{d}\theta=\alpha\theta\pm\beta\sin\theta\pm\lambda\sin 2\theta$	M1	3.1a
	$=400\left[11\theta+12\sin\theta+\sin 2\theta\right]$ or $=200\left[11\theta+12\sin\theta+\sin 2\theta\right]$	A1	1.1b
	Using limits $\theta=0$ and $\theta=\pi$ or $\theta=0$ and $\theta=2\pi$ as appropriate and subtracts the correct way round provided there is an attempt at integration $=400\left[11\pi-0\right]=4400\pi=13823.0\left(\text{cm}^2\right)$ or $=200\left[11\left(2\pi\right)-0\right]=4400\pi=13823.0\left(\text{cm}^2\right)$	M1	1.1b
	Volume $=$ area $\times 90=396\ 000\pi=1\ 244\ 070.691\left(\text{cm}^3\right)$	M1	3.4
	time = $\frac{1\,244\,070.691}{50\,000}=...$ or volume = 1244 litres therefore time = $\frac{1244}{50}=...$	M1	2.2b
	25 (minutes)	A1	3.2a
		(7)	
(d)	For example Polar equation is not likely to be accurate. Some comment that the sides will not be smooth and draws an appropriate conclusion. The hole may not be uniform depth The pond may leak/ ground may absorb some water	B1	3.5b
		(1)	
			(14 marks)

Notes:

(a)

M1: Complete method to find the correct form for $\frac{\mathrm{d}x}{\mathrm{d}\theta}$

A1: Correct $\frac{\mathrm{d}x}{\mathrm{d}\theta}$

M1: Sets $\frac{\mathrm{d}x}{\mathrm{d}\theta}=0$ and factorises to find values for either $\sin\theta$ or $\cos\theta$.

B1: Deduces that as $\sin\theta=0$ this provides two tangents. This can be implied by 2 values for θ

A1*: Concludes that as $\cos\theta=-\frac{p}{4}>-1\Rightarrow p<4$ and p is a positive constant $\therefore\ 0<p<4$

(b) **B1:** Correct shape and position.
(c) **M1:** Uses the model to find the area of the cross section $2\times\frac{1}{2}\int_0^{\pi}\left[20\left(3+2\cos\theta\right)\right]^2\,\mathrm{d}\theta$ or $\frac{1}{2}\int_0^{2\pi}\left[20\left(3+2\cos\theta\right)\right]^2\,\mathrm{d}\theta$ **M1:** Uses the identity $\cos 2\theta = 2\cos^2\theta - 1$ to integrate to the required form. **A1:** Correct integration. **M1:** Uses limits $\theta = 0$ and $\theta = \pi$ or $\theta = 0$ and $\theta = 2\pi$ as appropriate and subtracts the correct way around provided there is an attempt at integration. Note if first M1 is not awarded for incorrect limits then award this mark for their limits used. **M1:** Multiplies their area by 90 (cm). **M1**: Divides their volume by 50000 **A1:** 25 (minutes)
(d) **B1:** See scheme for examples. Any reference to the flow of water is B0

7.

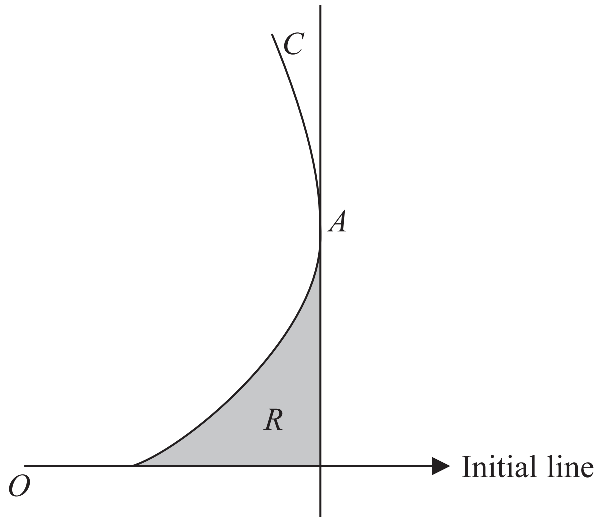


Figure 1

Figure 1 shows a sketch of the curve C with equation

$$r = 1 + \tan\theta \qquad\qquad 0 \leqslant \theta < \frac{\pi}{3}$$

Figure 1 also shows the tangent to C at the point A.
This tangent is perpendicular to the initial line.

(a) Use differentiation to prove that the polar coordinates of A are $\left(2, \frac{\pi}{4}\right)$

(4)

The finite region R, shown shaded in Figure 1, is bounded by C, the tangent at A and the initial line.

(b) Use calculus to show that the exact area of R is $\frac{1}{2}(1 - \ln 2)$

(6)

Question	Scheme	Marks	AOs
7(a)	$x = r\cos\theta = (1+\tan\theta)\cos\theta = \cos\theta + \sin\theta$ $= \cos\theta + \tan\theta\cos\theta$ $\frac{dx}{d\theta} = \alpha(1+\tan\theta)\sin\theta + \beta\sec^2\theta\cos\theta$ or $\frac{dx}{d\theta} = \alpha\sin\theta + \beta\cos\theta$ $\frac{dx}{d\theta} = \alpha\sin\theta + \beta\sec^2\theta\cos\theta + \delta\tan\theta\sin\theta$	M1	3.1a
	$\frac{dx}{d\theta} = -(1+\tan\theta)\sin\theta + \sec^2\theta\cos\theta$ or $\frac{dx}{d\theta} = -\sin\theta + \cos\theta$ $\frac{dx}{d\theta} = -\sin\theta + \sec^2\theta\cos\theta - \tan\theta\sin\theta$ or $\frac{dx}{d\theta} = -\sin\theta + \sec\theta - \tan\theta\sin\theta$	A1	1.1b
	For example $\left\{\frac{dx}{d\theta} =\right\} -\sin\theta + \cos\theta = 0 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \ldots$ $\left\{\frac{dx}{d\theta} =\right\} -\sin\theta + \cos\theta = 0 \Rightarrow \sin\theta = \cos\theta \Rightarrow \theta = \ldots$ $\left\{\frac{dx}{d\theta} =\right\} -\sin\theta + \cos\theta = \sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right) = \theta = \ldots$ or $\left\{\frac{dx}{d\theta} =\right\} -(1+\tan\theta)\sin\theta + \sec^2\theta\cos\theta = 0$ $\Rightarrow -\sin\theta - \frac{\sin^2\theta}{\cos\theta} + \frac{1}{\cos\theta} = 0 \Rightarrow -\sin\theta + \frac{1-\sin^2\theta}{\cos\theta} = 0$ $\Rightarrow -\sin\theta + \cos\theta = 0 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \ldots$ or $\left\{\frac{dx}{d\theta} =\right\} -\sin\theta - \tan\theta\sin\theta + \sec\theta = 0$ $\Rightarrow -\frac{1}{2}\sin 2\theta - \sin^2\theta + 1 = 0 \Rightarrow \sin 2\theta + 2\sin^2\theta - 1 = 1$ $\Rightarrow \sin 2\theta - \cos 2\theta = 1 \Rightarrow \sqrt{2}\sin\left(2\theta - \frac{\pi}{4}\right) = 1 \Rightarrow \theta = \ldots$ or $\left\{\frac{dx}{d\theta} =\right\} -\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = 0$ $\left\{\frac{dx}{d\theta} =\right\} -\left(1+\tan\left(\frac{\pi}{4}\right)\right)\sin\left(\frac{\pi}{4}\right) + \sec^2\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = 0$ $\left\{\frac{dx}{d\theta} =\right\} -\sin\left(\frac{\pi}{4}\right) + \sec^2\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) = 0$	dM1	3.1a
	$r = 1 + \tan\left(\frac{\pi}{4}\right) = 2$ therefore $A\left(2, \frac{\pi}{4}\right)$*	A1*	2.1
		(4)	
	Area bounded by the curve $= \frac{1}{2}\int(1+\tan\theta)^2 \ \{d\theta\}$	M1	3.1a

(b)	$= \frac{1}{2}\int (1 + 2\tan\theta + \tan^2\theta)\ \{d\theta\}$ $= \frac{1}{2}\int (1 + 2\tan\theta + [\sec^2\theta - 1])\ \{d\theta\} = \ldots$		
	$= \frac{1}{2}[2\ln\lvert\sec\theta\rvert + \tan\theta]$ or $\ln\lvert\sec\theta\rvert + \frac{1}{2}\tan\theta$ or $-\ln\cos\theta + \frac{1}{2}\tan\theta$ or $= \frac{1}{2}[-2\ln\lvert\cos\theta\rvert + \tan\theta]$	A1	1.1b
	$= \frac{1}{2}\left[2\ln\left\lvert\sec\left(\frac{\pi}{4}\right)\right\rvert + \tan\left(\frac{\pi}{4}\right)\right] - \frac{1}{2}[2\ln\lvert\sec(0)\rvert + \tan(0)]$ $= \left(\ln\left\lvert\sec\left(\frac{\pi}{4}\right)\right\rvert + \frac{1}{2}\tan\left(\frac{\pi}{4}\right)\right) - \left(\ln\lvert\sec 0\rvert + \frac{1}{2}\tan 0\right)$ $\left\{= \ln\sqrt{2} + \frac{1}{2}\right\}$	dM1	1.1b
	Area of triangle $= \frac{1}{2}xy = \frac{1}{2}\left(2\cos\frac{\pi}{4}\right)\left(2\sin\frac{\pi}{4}\right) = \ldots\left\{\frac{1}{2}\times\sqrt{2}\times\sqrt{2} = 1\right\}$ The equation of the tangent is $r = \sqrt{2}\sec\theta$ then applies Area bounded of triangle $= \frac{1}{2}\int_0^{\frac{\pi}{4}}(\sqrt{2}\sec\theta)^2\ \{d\theta\}$	M1	1.1b
	Finds the required area = area of triangle – area bounded by the curve $= 1 - \left[\ln\sqrt{2} + \frac{1}{2}\right]$ May be seen within an integral $= \frac{1}{2}\int(\sqrt{2}\sec\theta)^2\ \{d\theta\} - \frac{1}{2}\int(1 + \tan\theta)^2\ \{d\theta\}$	M1	3.1a
	$= \frac{1}{2}(1 - \ln 2)$ * cso	A1*	2.1
		(6)	
	Alternative Area bounded by the curve $= \frac{1}{2}\int(1 + \tan\theta)^2\ \{d\theta\}$ $= \frac{1}{2}\int(1 + 2\tan\theta + \tan^2\theta)\ \{d\theta\}$ let $u = \tan\theta \Rightarrow \frac{du}{d\theta} = \sec^2\theta$ Leading to $= \frac{1}{2}\int \frac{(1 + 2u + u^2)}{1 + u^2}\{du\} = \frac{1}{2}\int 1 + \frac{2u}{1 + u^2}\{du\} = \ldots$	M1	3.1a
	$\frac{1}{2}[u + \ln(1 + u^2)]$	A1	1.1b
	$\frac{1}{2}[(1 + \ln(1 + (1)^2)) - (0 + \ln 1)]$ or $\frac{1}{2}\left[\left(\tan\left(\frac{\pi}{4}\right) + \ln\left(1 + \tan^2\left(\frac{\pi}{4}\right)\right)\right) - (\tan(0) + \ln(1 + \tan^2(0)))\right]$ $\left\{= \frac{1}{2}\ln 2 + \frac{1}{2}\right\}$	dM1	1.1b
	Area of triangle $= \frac{1}{2}xy = \frac{1}{2}\left(2\cos\frac{\pi}{4}\right)\left(2\sin\frac{\pi}{4}\right) = \ldots\left\{\frac{1}{2}\times\sqrt{2}\times\sqrt{2} = 1\right\}$	M1	1.1b

	Finds the required area = area of triangle – area bounded by the curve $$= 1 - \left[ln\sqrt{2} + \frac{1}{2}\right]$$	M1	3.1a
	$= \frac{1}{2}(1 - ln\,2)$ *	A1*	2.1
		(6)	
			(10 marks)

Notes:

(a)

M1: Substitutes the equation of C into $x = r\,cos\,\theta$and differentiates to the required form

A1: Fully correct differentiation

dM1: Dependent on previous method mark. Sets their $\frac{dx}{d\theta} = 0$ and uses correct trig identities to find a value for θ. Alternatively substitutes $\theta = \frac{\pi}{4}$ into their $\frac{dx}{d\theta}$ and shows equals 0.

A1*: Shows that r = 2 and hence the polar coordinates $\left(2,\ \frac{\pi}{4}\right)$ from correct working

(b)

M1: Applies area $= \frac{1}{2}\int r^2\,\theta\ d\theta$, multiplies out, uses the identity $\pm 1 \pm tan^2\,\theta = sec^2\,\theta$to get into an integrable form **and** integrates. Condone missing $d\theta$, limits are not required for this mark

A1: Correct integration. Note may include θ – θ if the one's were not cancelled earlier.

dM1: Dependent on the first method mark. Applies the limits of $\theta = 0$ and $\theta = \frac{\pi}{4}$ and subtracts the correct way round. Since substitution of the limit $\theta = 0$is 0 so may be implied

M1: Correct method to find the area of triangle seen. This may be minimal but area = 1 only is M0, they need to show some method.

M1: Finds the required area = area of triangle – area bounded by the curve

A1*: Correct answer, with no errors or omissions. cso

Alternative

M1: Applies area $= \frac{1}{2}\int r^2\,\theta\ d\theta$, multiplies out, uses the substitution $u = tan\,\theta$ to get into an integrable form **and** integrates. Limits are not required for this mark

A1: Correct integration

dM1: Dependent on the first method mark. Applies the limits of u = 0 and u = 1 or substitutes back using $u = tan\,\theta$ and uses the limits $\theta = 0$ and $\theta = \frac{\pi}{4}$ and subtracts the correct way round. Since substitution of the limit $\theta = 0$is 0 so may be implied

M1: Correct method to find the area of triangle

M1: Finds the required area = area of triangle – area bounded by the curve

A1*: Correct answer, with no errors or omissions. cso