Cp2Ch4 XMQs and MS

(Total: 57 marks)

7.

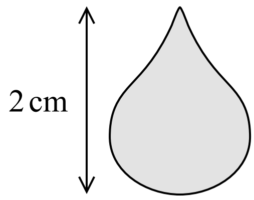

Figure 2

Figure 2 shows the image of a gold pendant which has height 2 cm. The pendant is modelled by a solid of revolution of a curve C about the y-axis. The curve C has parametric equations

$$x = \cos\theta + \frac{1}{2}\sin 2\theta, \quad y = -(1 + \sin\theta) \quad 0 \leqslant \theta \leqslant 2\pi$$

(a) Show that a Cartesian equation of the curve C is

$$x^2 = -(y^4 + 2y^3)$$

(4)

(b) Hence, using the model, find, in cm^3, the volume of the pendant.

(4)

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Question	Scheme	Marks	AOs
7(a)	$x = \cos\theta + \sin\theta\cos\theta = -y\cos\theta$	M1	2.1
	$\sin\theta = -y-1$	M1	2.1
	$\left(\frac{x}{-y}\right)^2 = 1-(-y-1)^2$	M1	2.1
	$x^2 = -\left(y^4 + 2y^3\right)$*	A1*	1.1b
		(4)	
(b)	$V = \pi\int x^2\,\mathrm{d}y = \pi\int -\left(y^4+2y^3\right)\,\mathrm{d}y$	M1	3.4
	$= \pi\left[-\left(\frac{y^5}{5}+\frac{y^4}{2}\right)\right]$	A1	1.1b
	$= -\pi\left[\left(\frac{(0)^5}{5}+\frac{(0)^4}{2}\right)-\left(\frac{(-2)^5}{5}+\frac{(-2)^4}{2}\right)\right]$	M1	3.4
	$= 1.6\pi\,\text{cm}^3$ **or** awrt 5.03 cm^3	A1	1.1b
		(4)	
			(8 marks)

Notes:

(a)

M1: Obtains x in terms of y and $\cos\theta$

M1: Obtains an equation connecting y and $\sin\theta$

M1: Uses Pythagoras to obtain an equation in x and y only

A1*: Obtains printed answer

(b)

M1: Uses the correct volume of revolution formula with the given expression

A1: Correct integration

M1: Correct use of correct limits

A1: Correct volume

7.

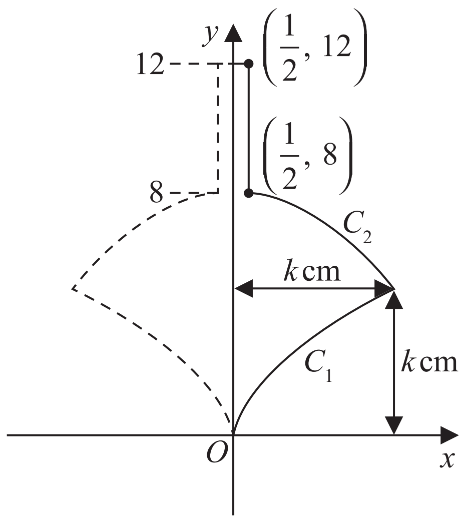

Diagram not to scale

Figure 2

Figure 2 shows a sketch of the cross-section of a design for a child's spinning top. The top is formed by rotating the region bounded by the y-axis, the curve C_1, the curve C_2, the line with equation $x = \frac{1}{2}$ and the line with equation $y = 12$, through 360° about the y-axis.

The curve C_1 has equation

$$y = k^{\frac{2}{3}}x^{\frac{1}{3}} \qquad 0 \leqslant x \leqslant k$$

and the curve C_2 has equation

$$y = \frac{32k^2 - k - (32 - 4k)x^2}{4k^2 - 1} \qquad \frac{1}{2} \leqslant x \leqslant k$$

(a) Show that $\displaystyle\int_k^8 \left((4k^2 - 1)y - (32k^2 - k)\right)\mathrm{d}y = \frac{1}{2}(8 - k)\left(4k^3 - 32k^2 + k - 8\right)$

(3)

Hence find

(b) the value of k that gives the maximum value for the volume of the spinning top,

(9)

(c) the maximum volume of the spinning top.

(3)

Question	Scheme	Marks	AOs
7(a)	$\int_k^8 \left((4k^2-1)y-(32k^2-k)\right)\,dy = \left[(4k^2-1)\frac{y^2}{2}-(32k^2-k)y\right]_k^8$ $=(4k^2-1)\frac{8^2-k^2}{2}-(32k^2-k)(8-k)$	M1	1.1b
	$=\frac{1}{2}(4k^2-1)(8-k)(8+k)-(32k^2-k)(8-k)$ $=\frac{1}{2}(8-k)\left((4k^2-1)(8+k)-2(32k^2-k)\right)$	M1	3.1a
	$=\frac{1}{2}(8-k)\left(32k^2+4k^3-8-k-64k^2+2k\right)$ $=\frac{1}{2}(8-k)\left(4k^3-32k^2+k-8\right)$*	A1*	2.1
		(3)	
(b)	Uses $(\pi)\int x^2\,dy$ with both of the curves and adds the results (a complete method to find the volume of the main body piece).	B1	3.1a
	Attempts $\int x^2 dy = \int \frac{y^6}{k^4}\,dy = ..$	M1	1.1b
	So $(\pi)\int_0^k x^2\,dy = \frac{(\pi)k^3}{7}$	A1	2.2a
	Attempts second curve $\int x^2 dy = \int \frac{(4k^2-1)y-(32k^2-k)}{-(32-4k)}\,dy = ..$	M1	1.1b
	$(\pi)\int_k^8 x^2 dy = \frac{\frac{1}{2}(8-k)\left(4k^3-32k^2+k-8\right)}{-4(8-k)} = (\pi)\frac{1}{8}\left(8-k+32k^2-4k^3\right)$	M1	2.1
	So volume of body $=(\pi)\left(\frac{k^3}{7}+\frac{1}{8}\left(8-k+32k^2-4k^3\right)\right)$ Or total volume $=(\pi)\left(1+\frac{k^3}{7}+\frac{1}{8}\left(8-k+32k^2-4k^3\right)\right)$	A1	1.1b
	$\frac{dV}{dk}=0 \Rightarrow \frac{3k^2}{7}+\frac{1}{8}\left(1+64k-12k^2\right)=0$	M1	3.1a
	$\Rightarrow 60k^2-448k+7=0 \Rightarrow k=...$	M1	1.1b
	But $k > ½$ so must be k = awrt 7.45 cm	A1	3.2a
		(9)	

(c)	Volume of handle is $\pi r^2 h$ $(= \pi(0.5)^2 \times 4) = \pi$	B1	2.2a
	So volume of spinning top is $V = \pi\left(1 + \frac{(7.45)^3}{7} + \frac{1}{8}\left(8 - (7.45) + 32(7.45)^2 - 4(7.45)^3\right)\right) = ...$	M1	1.1b
	= awrt 237 cm^3 (3 s.f.)	A1	1.1b
		(3)	
			(15 marks)

Notes:

(a)

M1: Correct attempt at integration and applies the limits.

M1: Applies completion of the square or expanding and factorising to obtain the factor $(k - 8)$ and removes this factor.

A1*: Correct completion, expands and collects terms inside the bracket. No errors seen and sufficient steps must be shown.

(b)

B1: Realises the needed to find the volume and attempts the formula at for both curves, adding the result. Note the π is not necessary at all for part (b).

M1: Attempts to make x^2 the subject of the first equation and attempts to integrate it. Power of k may be incorrect.

A1: Correct limits 0 and k applied to deduce the volume in terms of k for this section.

M1: Attempts the integral for top portion of body, make x^2 the subject, including the $32 – 4k$ in the denominator.

M1: Obtains the result using (a) and cancels the $(8 - k)$ term to achieve a cubic in k.

A1: Adds the results of the integrals to give the volume for the whole spinning top, or just the body. If the volume of the cylinder is incorrect, ignore this term for the accuracy – the non-constant terms should all be correct.

M1: Realises the need to differentiate the result and set equal to 0 to obtain the x value at any stationary points. Must be a valid attempt at differentiating.

M1: Solves the quadratic (usual rules).

A1: Correct answer, with second root rejected, or comment why this root gives the maximum.

(c) Allow marks for part (c) for the volume if the work is done in part (b).

B1: Deduces correct volume π for the handle. Cylinder formula or use of integration may be used. Award wherever seen - could be in (b).

M1: Substitutes their value for k into their volume formula. Dependent on the volume formula having been the sum of the three sections and including the factor π - handle must be included. May have been done in (b).

A1: Awrt 237cm^3 NB if this is given in (b) allow this mark unless a different answer is given in (c), in which case count the answer given in (c) as their answer.

8.

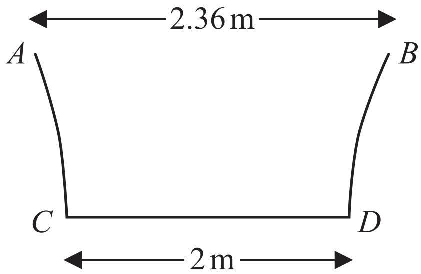

Figure 1

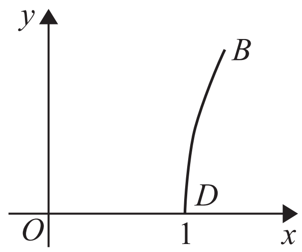

Figure 2

Figure 1 shows the central vertical cross section *ABCD* of a paddling pool that has a circular horizontal cross section. Measurements of the diameters of the top and bottom of the paddling pool have been taken in order to estimate the volume of water that the paddling pool can contain.

Using these measurements, the curve *BD* is modelled by the equation

$$y = \ln(3.6x - k) \qquad 1 \leqslant x \leqslant 1.18$$

as shown in Figure 2.

(a) Find the value of *k*.

(1)

(b) Find the depth of the paddling pool according to this model.

(2)

The pool is being filled with water from a tap.

(c) Find, in terms of *h*, the volume of water in the pool when the pool is filled to a depth of *h* m.

(5)

Given that the pool is being filled at a constant rate of 15 litres every minute,

(d) find, in cm h^{-1}, the rate at which the water level is rising in the pool when the depth of the water is 0.2 m.

(3)

Question	Scheme	Marks	AOs
8(a)	$k = 2.6$	B1	3.4
		(1)	
(b)	$x = 1.18 \Rightarrow \ln\left(3.6\times1.18 - "2.6"\right) = ...$	M1	1.1b
	$h = 0.4995...$ m	A1	2.2b
		(2)	
(c)	$y = \ln\left(3.6x - 2.6\right) \Rightarrow x = \dfrac{e^y + 2.6}{3.6}$ or $\dfrac{5e^y + 13}{18}$	B1ft	1.1a
	$V = \pi\int\left(\dfrac{e^y + 2.6}{3.6}\right)^2 dy = \dfrac{\pi}{3.6^2}\int\left(e^{2y} + 5.2e^y + 6.76\right)dy$ or $\dfrac{\pi}{324}\int\left(25e^{2y} + 130e^y + 169\right)dy$	M1	3.3
	$= \dfrac{\pi}{3.6^2}\left[\dfrac{1}{2}e^{2y} + 5.2e^y + 6.76y\right]$ $\left(\text{or } \dfrac{\pi}{324}\left[\dfrac{25}{2}e^{2y} + 130e^y + 169y\right]\right)$	A1	1.1b
	$= \dfrac{\pi}{3.6^2}\left\{\left(\dfrac{1}{2}e^{2h} + 5.2e^h + 6.76h\right) - \left(\dfrac{1}{2}e^0 + 5.2e^0 + 6.76(0)\right)\right\}$ or e.g. $= \dfrac{\pi}{324}\left\{\left(\dfrac{25}{2}e^{2h} + 130e^h + 169h\right) - \left(\dfrac{25}{2}e^0 + 130e^0 + 6.76(0)\right)\right\}$	M1	2.1
	$= \dfrac{\pi}{3.6^2}\left(\dfrac{1}{2}e^{2h} + 5.2e^h + 6.76h - 5.7\right)$	A1	1.1b
		(5)	
(d)	$\dfrac{dV}{dh} = \dfrac{\pi}{3.6^2}\left(e^{2h} + 5.2e^h + 6.76\right) = \dfrac{\pi}{3.6^2}\left(e^{0.4} + 5.2e^{0.2} + 6.76\right)$	M1	3.1a
	$\dfrac{dh}{dt} = \dfrac{dh}{dV}\dfrac{dV}{dt} = \dfrac{1}{3.539...}\times 0.015\times 60$	M1	1.1b
	$\dfrac{dh}{dt} = 25.4\,\text{cm h}^{-1}$	A1	3.2a
		(3)	
(d) Way 2	$y = 0.2 \Rightarrow x = \dfrac{2.6 + e^{0.2}}{3.6} \Rightarrow A = \pi\left(\dfrac{2.6 + e^{0.2}}{3.6}\right)^2 (= 3.54)$	M1	3.1a
	$\dfrac{dh}{dt} = \dfrac{0.015\times 60}{3.54}$	M1	1.1b
	$\dfrac{dh}{dt} = 25.4\,\text{cm h}^{-1}$	A1	3.2a
			(11 marks)

Notes

(a)
B1: Uses the model to obtain a correct value for *k*. Must be 2.6 not -2.6
(b)

M1: Substitutes their value of k and $x = 1.18$ into the given model to find a value for y
A1: Infers that the depth of the pool could be awrt 0.5 m
(c)
B1ft: Uses the model to obtain x correctly in terms of y (follow through their k)
M1: Uses the model to obtain an expression for the volume of the pool using $\pi\int\left(\textit{their } f(y)\right)^2 \, \mathrm{d}y$ – must expand in order to reach an integrable form (allow poor squaring e.g. $(a + b)^2 = a^2 + b^2$. **Note that the π may be recovered later.**
A1: Correct integration
M1: Selects limits appropriate to the model (h and 0) substitutes and clearly shows the use of both limits (i.e. including zero)
A1: Correct expression **(allow unsimplified and isw if necessary)**
(d)
Way 1
M1: Recognises that $\frac{\mathrm{d}V}{\mathrm{d}h}$ is required and attempts to find $\frac{\mathrm{d}V}{\mathrm{d}h}$ or $\frac{\mathrm{d}h}{\mathrm{d}V}$ from their integration or using the earlier result (before integrating). Must clearly be identified as $\frac{\mathrm{d}V}{\mathrm{d}h}$ or $\frac{\mathrm{d}h}{\mathrm{d}V}$ unless this implied by subsequent work.
M1: Evidence of the correct use of the chain rule (ignore any confusion with units). Look for an attempt to divide 15 or their converted 15 by their $\frac{\mathrm{d}V}{\mathrm{d}h}$ or to multiply 15 or their converted 15 by $\frac{\mathrm{d}h}{\mathrm{d}V}$ **but must reach a <u>value</u> for $\frac{\mathrm{d}h}{\mathrm{d}t}$ <u>but you do not need to check their value.</u>**
A1: Interprets their solution correctly to obtain the correct answer (awrt 25.4) **with the correct units**
Way 2
M1: Uses $y = 0.2$ to find x and the surface area of the water at that instant
M1: Attempts to divide the rate by their area (ignore any confusion with units)
A1: Interprets their solution correctly to obtain the correct answer (awrt 25.4) **with the correct units**

7.

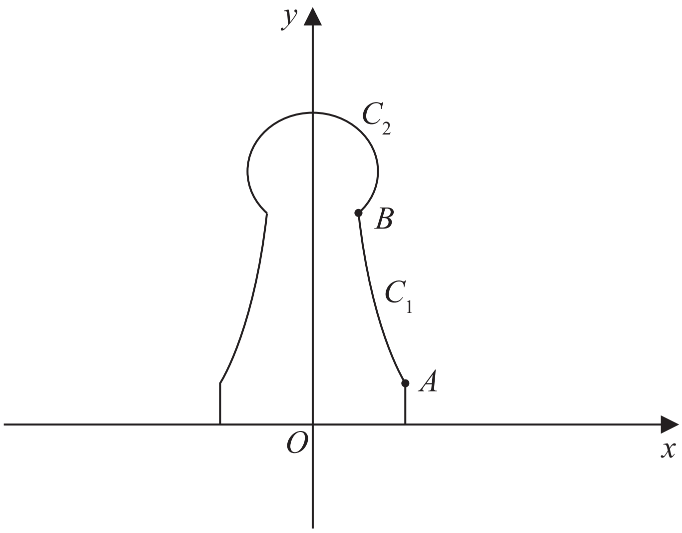

Figure 1

A student wants to make plastic chess pieces using a 3D printer. Figure 1 shows the central vertical cross-section of the student's design for one chess piece. The plastic chess piece is formed by rotating the region bounded by the y-axis, the x-axis, the line with equation $x = 1$, the curve C_1 and the curve C_2 through 360° about the y-axis.

The point A has coordinates (1, 0.5) and the point B has coordinates (0.5, 2.5) where the units are centimetres.

The curve C_1 is modelled by the equation

$$x = \frac{a}{y + b} \qquad 0.5 \leqslant y \leqslant 2.5$$

(a) Determine the value of a and the value of b according to the model.

(2)

The curve C_2 is modelled to be an arc of the circle with centre (0, 3).

(b) Use calculus to determine the volume of plastic required to make the chess piece according to the model.

(9)

Question	Scheme	Marks	AOs
7(a)	$1=\dfrac{a}{0.5+b},\ 0.5=\dfrac{a}{2.5+b}\Rightarrow a=...,\ b=...$	M1	3.3
	$a = 2$, $b = 1.5$	A1	1.1b
		(2)	
(b)	$V_1=\pi\int x^2\mathrm{d}y=\pi\int\left(\dfrac{"2"}{y+"1.5"}\right)^2\mathrm{d}y$	B1ft	3.4
	$\pi\int_{0.5}^{2.5}\left(\dfrac{"2"}{y+"1.5"}\right)^2\mathrm{d}y$	M1	1.1a
	$=\{4\pi\}\left[-(y+1.5)^{-1}\right]_{0.5}^{2.5}(=\pi)$	M1	1.1b
	$x^2+(y-3)^2=0.5$	B1	2.2a
	$V_2=\pi\int x^2\mathrm{d}y=\pi\int\left(0.5-(y-3)^2\right)\mathrm{d}y$ or $\pi\int\left(-y^2+6y-8.5\right)\mathrm{d}y$	M1	1.1b
	$=\pi\int_{2.5}^{3+\frac{1}{\sqrt{2}}}\left(0.5-(y-3)^2\right)\mathrm{d}y$ or $=\pi\int_{2.5}^{3+\frac{1}{\sqrt{2}}}\left(-y^2+6y-8.5\right)\mathrm{d}y$	M1	3.3
	$=\{\pi\}\left[0.5y-\frac{1}{3}(y-3)^3\right]_{2.5}^{3+\frac{1}{\sqrt{2}}}$ or $=\{\pi\}\left[-\frac{1}{3}y^3+3y^2-8.5y\right]_{2.5}^{3+\frac{1}{\sqrt{2}}}$	A1	1.1b
	$V_1+V_2+\text{cylinder}=\pi+\pi\left(\dfrac{5}{24}+\dfrac{\sqrt{2}}{6}\right)+\dfrac{1}{2}\pi$	dM1	3.4
	$=\pi\left(\dfrac{41}{24}+\dfrac{\sqrt{2}}{6}\right)\approx 6.11\text{cm}^3$	A1	2.2b
		(9)	
			(11 marks)

Notes

(a)
M1: Uses the given coordinates correctly in the equation modelling the curve to obtain at least one correct equation and attempts to find the values of *a* and *b*
A1: Correct values
(b)
B1ft: Uses the model to obtain $\pi\int\left(\dfrac{\text{their }a}{y+\text{their }b}\right)^2\mathrm{d}y$. Note the p can be recovered if appears later.
M1: Chooses limits appropriate to the model i.e. 0.5 and 2.5
M1: Integrates to obtain an expression of the form $k(y + \text{“1.5”})^{-1}$
B1: Deduces the correct equation for the circle
M1: Uses their circle equation and $\pi\int x^2\,\mathrm{d}y$ to attempt the top volume. Note the p can be recovered if appears later.

M1: Identifies limits appropriate to the model i.e. 2.5 and 3 + their radius
A1: Correct integration
dM1: Uses the model to find the volume of the chess piece including the cylindrical base (dependent on all previous method marks)
A1: Correct volume

8. (a) Given

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad n \in \mathbb{N}$$

show that

$$32\cos^6\theta \equiv \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$$

(5)

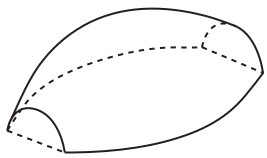

Figure 1

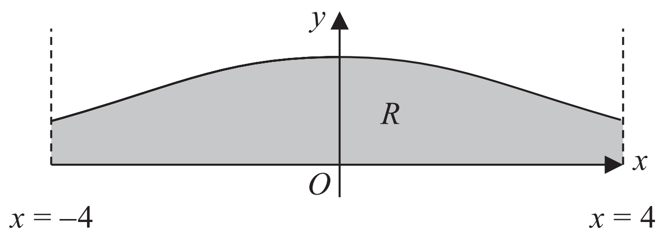

Figure 2

Figure 1 shows a solid paperweight with a flat base.

Figure 2 shows the curve with equation

$$y = H\cos^3\left(\frac{x}{4}\right) \qquad -4 \leqslant x \leqslant 4$$

where H is a positive constant and x is in radians.

The region R, shown shaded in Figure 2, is bounded by the curve, the line with equation $x = -4$, the line with equation $x = 4$ and the x-axis.

The paperweight is modelled by the solid of revolution formed when R is rotated **180°** about the x-axis.

Given that the maximum height of the paperweight is 2 cm,

(b) write down the value of H.

(1)

(c) Using algebraic integration and the result in part (a), determine, in cm^3, the volume of the paperweight, according to the model. Give your answer to 2 decimal places.

[*Solutions based entirely on calculator technology are not acceptable.*]

(5)

(d) State a limitation of the model.

(1)

Question	Scheme	Marks	AOs
8(a)	$\left(z+\frac{1}{z}\right)^6 = 64\cos^6\theta$	B1	2.1
	$\left(z+\frac{1}{z}\right)^6 = z^6 + 6(z^5)\left(\frac{1}{z}\right) + 15(z^4)\left(\frac{1}{z^2}\right) + 20(z^3)\left(\frac{1}{z^3}\right) + 15(z^2)\left(\frac{1}{z^4}\right) + 6(z)\left(\frac{1}{z^5}\right) + \left(\frac{1}{z^6}\right)$	M1	2.1
	$= \left[z^6 + \frac{1}{z^6}\right] + 6\left[z^4 + \frac{1}{z^4}\right] + 15\left[z^2 + \frac{1}{z^2}\right] + 20$	A1	1.1b
	Uses $z^n + \frac{1}{z^n} = 2\cos n\theta$ $\{64\cos^6\theta\} = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$	M1	2.1
	$32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$ * cso	A1 *	1.1b
		(5)	
(b)	$\boldsymbol{H = 2}$	B1	3.3
		(1)	
(c)	$\text{vol} = \left\{\frac{1}{2}\right\}\pi\int\left(2\cos^3\left(\frac{x}{4}\right)\right)^2 \mathrm{d}x$	B1ft	3.4
	$\text{vol} = \{2\pi\}\int\cos^6\left(\frac{x}{4}\right)\mathrm{d}x$ $= \{2\pi\}\int\frac{1}{32}\left(\cos\left(\frac{6x}{4}\right) + 6\cos\left(\frac{4x}{4}\right) + 15\cos\left(\frac{2x}{4}\right) + 10\right)\mathrm{d}x = \ldots$	M1	1.1b
	$= \{2\pi\}\left[\frac{1}{32}\left(\frac{2}{3}\sin\left(\frac{3x}{2}\right) + 6\sin(x) + 30\sin\left(\frac{x}{2}\right) + 10x\right)\right]$	A1	1.1b
	$= 2\times 2\pi\left[\frac{1}{32}\left(\frac{2}{3}\sin\left(\frac{3}{2}\times 4\right) + 6\sin(4) + 30\sin\left(\frac{4}{2}\right) + (10\times 4)\right) - 0\right] = \ldots$ or $=$ $2\pi\left[\frac{1}{32}\left(\frac{2}{3}\sin\left(\frac{3}{2}\times 4\right) + 6\sin(4) + 30\sin\left(\frac{4}{2}\right) + (10\times 4)\right) - \frac{1}{32}\left(\frac{2}{3}\sin\left(\frac{3}{2}\times -4\right) + 6\sin(-4) + 30\sin\left(-\frac{4}{2}\right) + (10\times -4)\right)\right]$ …	dM1	3.4
	= 24.56	A1	1.1b
		(5)	

(d)	The equation of the curve may not be suitable The measurements may not be accurate The paperweight may not be smooth	B1	3.5b
		(1)	
			(12 marks)

Notes:

(a)
B1: Correct identity or equivalent rearrangement. This can appear anywhere in the proof.
M1: Attempts the expansion of $\left(z + \frac{1}{z}\right)^6$ must have at least 3 correct terms. Combining the powers when expanding is fine.
A1: Correct expansion with *z* terms simplified, need not be rearranged. (So a correct expansion will score M1A1.)
M1: Uses $z^n + \frac{1}{z^n} = 2\cos n\,\theta$ to write the expression in terms multiple angles of $\cos 6\,\theta,\ \cos 4\,\theta$ and $\cos 2\,\theta$. Pairing of terms must be seen.
A1*: Achieves the printed answer with no errors or omissions. Cso
For approaches using De Moivre B0M1A1M0A0 may be scored if the binomial expansions is attempted (and correct for the A).

(b)
B1: See scheme

(c)
Note: The question instructs use of algebraic integration and part (a), so answer only can score at most B1 for implied correct formula.
B1ft: Correct expression for the volume of the paperweight **or** the solid formed through 360º rotation, stated or implied, ignore limits. No need to expand, but must be applied, not just a formula in *y,* though allow a correct formula followed by correct integral if the π disappears. Follow through their *H*
M1: Uses the result in part part (a) to express the volume in an integrable form and attempts to integrate. Note use of *θ* instead of *x* is permissible for this mark. Allow if one term is missing or miscopied.
A1: Correct integration in terms of *x*. Ignore π, their H^2 and the $\frac{1}{32}$. Note if *θ* has been used it is A0 unless a correct substitution method has been implied as the coefficients will be incorrect.
dM1: Dependent on previous method mark and must have reached and integral of the correct form - - in terms of *x* with correct arguments allowing for one slip. Finds the required volume using either$\pi\int_0^4 y^2\,\mathrm{d}x$ or $\frac{1}{2}\pi\int_{-4}^4 y^2\,\mathrm{d}x$ and applies their limits - accept any value following a valid attempt at the integration as an attempt at applying limits.
A1: cao 24.56

(d)
B1: States an appropriate limitation. See scheme for some examples. The limitation should refer to the paperweight, not to paper. Do not accept "it does not take into account thickness of material" as it is a solid, not a shell, being modelled. Award the mark for a correct reason if two reasons are given and one is incorrect.