Cp2Ch1 XMQs and MS

(Total: 60 marks)

1. CP2_Sample Q4 . 7 marks - CP2ch1 Complex numbers
2. CP2_Specimen Q4 . 7 marks - CP2ch1 Complex numbers
3. CP2_2019 Q4 . 8 marks - CP2ch1 Complex numbers
4. CP2_2020 Q4 . 10 marks - CP2ch1 Complex numbers
5. CP2_2021 Q8 . 11 marks - CP2ch1 Complex numbers
6. CP2_2021 Q9 . 8 marks - CP2ch1 Complex numbers
7. CP2_2022 Q1 . 3 marks - CP1ch2 Argand diagrams
8. CP2_2022 Q4 . 6 marks - CP2ch1 Complex numbers

4. A complex number z has modulus 1 and argument θ.

(a) Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad n \in \mathbb{Z}^+$$

(2)

(b) Hence, show that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

(5)

Question	Scheme	Marks	AOs
4(a)	$z^n + z^{-n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta$	M1	2.1
	$= 2\cos n\theta$ *	A1*	1.1b
		(2)	
(b)	$\left(z + z^{-1}\right)^4 = 16\cos^4\theta$	B1	2.1
	$\left(z + z^{-1}\right)^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$	M1	2.1
	$= z^4 + z^{-4} + 4\left(z^2 + z^{-2}\right) + 6$	A1	1.1b
	$= 2\cos 4\theta + 4\left(2\cos 2\theta\right) + 6$	M1	2.1
	$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$*	A1*	1.1b
		(5)	
			(7 marks)

Notes:

(a)

M1: Identifies the correct form for z^n and z^{-n} and adds to progress to the printed answer

A1*: Achieves printed answer with no errors

(b)

B1: Begins the argument by using the correct index with the result from part (a)

M1: Realises the need to find the expansion of $\left(z + z^{-1}\right)^4$

A1: Terms correctly combined

M1: Links the expansion with the result in part (a)

A1*: Achieves printed answer with no errors

4. (a) Using the identity $zz^* = |z|^2$, or otherwise, show that if w is any root of unity then

$$|w - 2|^2 = 5 - 2(w + w^*)$$

(3)

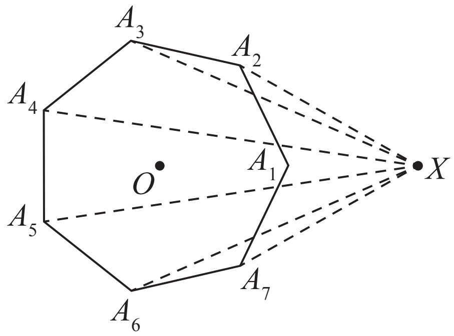

Figure 1

Figure 1 shows a regular heptagon $A_1A_2A_3A_4A_5A_6A_7$ whose vertices all lie on the unit circle with centre at the origin O and A_1 at (1, 0). The point X lies in the same plane as the heptagon and has coordinates (2, 0).

Using the result given in part (a),

(b) find $\sum_{i=1}^{7} (XA_i)^2$

(4)

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Question	Scheme	Marks	AOs
4(a)	$\|w-2\|^2 = (w-2)(w-2)^* = (w-2)(w^*-2)$	M1	1.1b
	$= ww^* - 2w - 2w^* + 4 = \|w\|^2 - 2(w+w^*) + 4$	M1	1.1b
	$= 1+4-2(w+w^*) = 5-2(w+w^*)$ since w is a root of unity so has modulus 1. *	A1*	2.1
		(3)	
Alt	$w = x+\mathrm{i}y \Rightarrow \|w-2\|^2 = \|(x-2)+\mathrm{i}y\|^2 = (x-2)^2+y^2$	M1	1.1b
	$= x^2-4x+4+y^2 = x^2+y^2+4-2(x+\mathrm{i}y+x-\mathrm{i}y)$	M1	1.1b
	$=1+4+2(w+w^*)$ since $x^2+y^2=1$ as w is a root of unity. *	A1*	2.1
		(3)	
(b)	$\sum_{i=1}^{7}(XA_i)^2 = \sum_{i=1}^{7}\|w_i-2\|^2$ where w_i are the 7th roots of unity.	M1	3.1a
	$=\sum_{i=1}^{7}\left(5-2(w_i+w_i^*)\right) = \sum_{i=1}^{7}5 - 2\sum_{i=1}^{7}(w_i+w_i^*)$	M1	1.1b
	$\sum_{i=1}^{7}(w_i+w_i^*)=0$ since roots of unity sum to zero.	B1	2.2a
	So $\sum_{i=1}^{7}(XA_i)^2 = 7\times5 = 35$	A1	1.1b
		(4)	
			(7 marks)

Notes:

(a)
M1: Uses the given identity and distributivity of the conjugate.
M1: Expands and collects terms
A1*: Completes the proof with justification of $|w| = 1$.
Alt
M1: Replaces w by $x + \mathrm{i}y$ and applied the modulus squared.
M1: Expands the brackets and gathers $x^2 + y^2$ (may be implied if $x^2+y^2=1$ stated explicitly) and splits the x term (may be implied if $w+w^*=2x$ stated explicitly).
A1*: Completes proof convincingly with justification for $x^2+y^2=1$ given.

(b)
M1: Makes the connection with part (a) and translates into a complex plane problem, realising the vertices lie at 7th roots of unity.
M1: Uses the identity shown in (a) and splits the sum.
B1: Deduces the second sum is zero as sum of roots of unity is zero.
A1: Correct answer.

4. The infinite series C and S are defined by

$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + \dots$$

$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + \dots$$

Given that the series C and S are both convergent,

(a) show that

$$C + iS = \frac{2e^{i\theta}}{2 - e^{4i\theta}}$$

(4)

(b) Hence show that

$$S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}$$

(4)

Question	Scheme	Marks	AOs
4(a) **Way 1**	$\mathrm{C}+\mathrm{iS}=\cos\theta+\mathrm{i}\sin\theta+\frac{1}{2}(\cos 5\theta+\mathrm{i}\sin 5\theta)\left(+\frac{1}{4}(\cos 9\theta+\mathrm{i}\sin 9\theta)+...\right)$	M1	1.1b
	$=\mathrm{e}^{\mathrm{i}\theta}+\frac{1}{2}\mathrm{e}^{5\mathrm{i}\theta}\left(+\frac{1}{4}\mathrm{e}^{9\mathrm{i}\theta}+...\right)$	A1	2.1
	$\mathrm{C}+\mathrm{iS}=\frac{\mathrm{e}^{\mathrm{i}\theta}}{1-\frac{1}{2}\mathrm{e}^{4\mathrm{i}\theta}}$	M1	3.1a
	$=\frac{2\mathrm{e}^{\mathrm{i}\theta}}{2-\mathrm{e}^{4\mathrm{i}\theta}}$ *	A1*	1.1b
		(4)	
(a) **Way 2**	$\mathrm{C}+\mathrm{iS}=\cos\theta+\mathrm{i}\sin\theta+\frac{1}{2}(\cos 5\theta+\mathrm{i}\sin 5\theta)\left(+\frac{1}{4}(\cos 9\theta+\mathrm{i}\sin 9\theta)+...\right)$	M1	1.1b
	$\mathrm{C}+\mathrm{iS}=\cos\theta+\mathrm{i}\sin\theta+\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)^5\left(+\frac{1}{4}(\cos\theta+\mathrm{i}\sin\theta)^9+...\right)$	A1	2.1
	$\mathrm{C}+\mathrm{iS}=\frac{\cos\theta+\mathrm{i}\sin\theta}{1-\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)^4}=\frac{\mathrm{e}^{\mathrm{i}\theta}}{1-\frac{1}{2}\mathrm{e}^{4\mathrm{i}\theta}}$	M1	3.1a
	$=\frac{2\mathrm{e}^{\mathrm{i}\theta}}{2-\mathrm{e}^{4\mathrm{i}\theta}}$ *	A1*	1.1b
		(4)	
(b) **Way 1**	$\frac{2\mathrm{e}^{\mathrm{i}\theta}}{2-\mathrm{e}^{4\mathrm{i}\theta}}\times\frac{2-\mathrm{e}^{-4\mathrm{i}\theta}}{2-\mathrm{e}^{-4\mathrm{i}\theta}}$	M1	3.1a
	$\frac{4\mathrm{e}^{\mathrm{i}\theta}-2\mathrm{e}^{-3\mathrm{i}\theta}}{4-2\mathrm{e}^{-4\mathrm{i}\theta}-2\mathrm{e}^{4\mathrm{i}\theta}+1}$	A1	1.1b
	$\frac{4\cos\theta+4\mathrm{i}\sin\theta-2\cos 3\theta+2\mathrm{i}\sin 3\theta}{5-2\cos 4\theta+2\mathrm{i}\sin 4\theta-2\cos 4\theta-2\mathrm{i}\sin 4\theta}$ **Dependent on the first M**	**d**M1	2.1
	$S=\frac{4\sin\theta+2\sin 3\theta}{5-4\cos 4\theta}$ *	A1*	1.1b
		(4)	
(b) **Way 2**	$\frac{2\mathrm{e}^{\mathrm{i}\theta}}{2-\mathrm{e}^{4\mathrm{i}\theta}}=\frac{2(\cos\theta+\mathrm{i}\sin\theta)}{2-(\cos 4\theta+\mathrm{i}\sin 4\theta)}\times\frac{2-(\cos 4\theta-\mathrm{i}\sin 4\theta)}{2-(\cos 4\theta-\mathrm{i}\sin 4\theta)}$	M1	3.1a
	$\frac{4\cos\theta+4\mathrm{i}\sin\theta-2\cos\theta\cos 4\theta-2\sin\theta\sin 4\theta+2\mathrm{i}\sin 4\theta\cos\theta-2\mathrm{i}\sin\theta\cos 4\theta}{4+\cos^2 4\theta+\sin^2 4\theta-4\cos 4\theta}$	A1	1.1b
	$\frac{4\cos\theta+4\mathrm{i}\sin\theta-2\cos 3\theta+2\mathrm{i}\sin 3\theta}{5-2\cos 4\theta+2\mathrm{i}\sin 4\theta-2\cos 4\theta-2\mathrm{i}\sin 4\theta}$ **Dependent on the first M**	**d**M1	2.1
	$S=\frac{4\sin\theta+2\sin 3\theta}{5-4\cos 4\theta}$ *	A1*	1.1b
	(8 marks)		

Notes
(a) **Way 1** M1: Combines the two series by pairing the multiples of θ (At least up to 5θ) A1: Converts to Euler form correctly (At least up to 5θ) M1: Recognises that C + iS is a convergent geometric series and uses the sum to infinity of a GP A1*: Reaches the printed answer with no errors **Way 2** M1: Combines the two series by pairing the multiples of θ (At least up to 5θ) A1: Converts to power form correctly (At least up to 5θ) M1: Recognises that C + iS is a convergent geometric series and uses the sum to infinity of a GP A1*: Reaches the printed answer with no errors (b) **Way 1** M1: Multiplies numerator and denominator by $2-\mathrm{e}^{-4\mathrm{i}\theta}$ A1: Correct fraction in terms of exponentials **d**M1: Converts back to trigonometric form A1*: Reaches the printed answer with no errors **Way 2** M1: Converts back to trigonometric form and realises the need to make the denominator real and multiplies numerator and denominator by the complex conjugate of the denominator which is **correct** for their fraction A1: Correct fraction in terms of trigonometric functions **d**M1: Uses the correct addition formula to obtain sin 3θ in the numerator A1*: Reaches the printed answer with no errors

4. (a) Use de Moivre's theorem to prove that

$$\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$$

(5)

(b) Hence find the distinct roots of the equation

$$1 + 7x - 56x^3 + 112x^5 - 64x^7 = 0$$

(5)

giving your answer to 3 decimal places where appropriate.

Question	Scheme	Marks	AOs
4(a)	$\left(\cos\theta+\mathrm{i}\sin\theta\right)^7=\cos^7\theta+\binom{7}{1}\cos^6\theta\left(\mathrm{i}\sin\theta\right)+\binom{7}{2}\cos^5\theta\left(\mathrm{i}\sin\theta\right)^2+...$ Some simplification may be done at this stage e.g. $c^7+7c^6\mathrm{i}s-21c^5s^2-35c^4\mathrm{i}s^3+35c^3s^4+21c^2\mathrm{i}s^5-7cs^6-\mathrm{i}s^7$	M1	1.1b
	$\mathrm{i}\sin 7\theta={}^7\mathrm{C}_1c^6\,\mathrm{i}s+{}^7\mathrm{C}_3c^4\,\mathrm{i}^3s^3+{}^7\mathrm{C}_5c^2\mathrm{i}^5s^5+\mathrm{i}^7s^7$ or $\quad=7c^6\,\mathrm{i}s+35c^4\,\mathrm{i}^3s^3+21c^2\mathrm{i}^5s^5+\mathrm{i}^7s^7$	M1	2.1
	$\sin 7\theta=7c^6s-35c^4s^3+21c^2s^5-s^7$	A1	1.1b
	$=7\left(1-s^2\right)^3s-35\left(1-s^2\right)^2s^3+21\left(1-s^2\right)s^5-s^7$ $=7\left(1-3s^2+3s^4-s^6\right)s-35\left(1-2s^2+s^4\right)s^3+21\left(1-s^2\right)s^5-s^7$	M1	2.1
	$\left\{7s-21s^3+21s^5-7s^7-35s^3+70s^5-35s^7+21s^5-21s^7-s^7\right\}$ leading to $\sin 7\theta=7\sin\theta-56\sin^3\theta+112\sin^5\theta-64\sin^7\theta$ *	A1*	1.1b
		(5)	
(b)	$1+\sin 7\theta=0\Rightarrow\sin 7\theta=-1$	M1	3.1a
	$7\theta=-450,-90,\ 270,\ 630,\ ...$ or $7\theta=-\frac{5\pi}{2},-\frac{\pi}{2},\frac{3\pi}{2},\frac{7\pi}{2},...$	A1	1.1b
	$\theta=-\frac{450}{7},-\frac{90}{7},\frac{270}{7},\frac{630}{7},...\Rightarrow\sin\theta=...$ or $\theta=-\frac{5\pi}{14},-\frac{\pi}{14},\frac{3\pi}{14},\frac{7\pi}{14},...\Rightarrow\sin\theta=...$	M1	2.2a
	$x=\sin\theta=-0.901,\ -0.223,\ 0.623,\ 1$	A1 A1	1.1b 2.3
		(5)	
			(10 marks)

Notes

(a)

M1: Attempts to expand $\left(\cos\theta+\mathrm{i}\sin\theta\right)^7$ including a recognisable attempt at binomial coefficients

Some simplification may be done at this stage. (May only see imaginary terms)

M1: Identifies imaginary terms with sin 7θ

A1: Correct expression with coefficients evaluated and i's dealt with correctly

M1: Replaces $\cos^2\theta$ with $1-\sin^2\theta$ and applies the expansions of $\left(1-\sin^2\theta\right)^2$ and $\left(1-\sin^2\theta\right)^3$ to their expression

A1*: Reaches the printed answer with no errors and expansion of brackets seen.

(b)

M1: Makes the connection with part (a) and realises the need to solve $\sin 7\theta=-1$

A1: At least one correct value for 7θ

M1: Divides by 7 and deduces that *x* values are found by finding at least one value for $\sin\theta$

A1: Awrt 2 correct values for *x*

A1: Awrt all 4 *x* values correct and no extras

8. (i) The point P is one vertex of a regular pentagon in an Argand diagram. The centre of the pentagon is at the origin.

Given that P represents the complex number $6 + 6\mathrm{i}$, determine the complex numbers that represent the other vertices of the pentagon, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$

(5)

(ii) (a) On a single Argand diagram, shade the region, R, that satisfies both

$$|z - 2\mathrm{i}| \leqslant 2 \qquad \text{and} \qquad \frac{1}{4}\pi \leqslant \arg z \leqslant \frac{1}{3}\pi$$

(2)

(b) Determine the exact area of R, giving your answer in simplest form.

(4)

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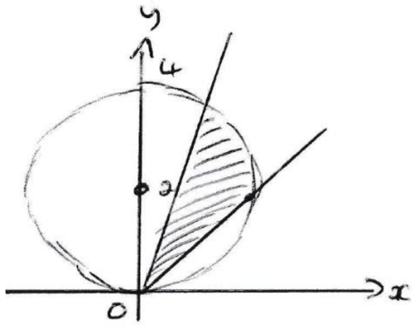
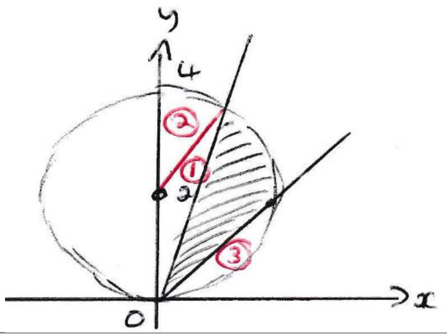
Question	Scheme	Marks	AOs
8(i)	$\lvert z\rvert=\sqrt{6^2+6^2}=\ldots\ 6\sqrt{2}$ or $\sqrt{72}$ and $\arg z=\tan^{-1}\left(\frac{6}{6}\right)=\ldots\left\{\frac{\pi}{4}\right\}$ Can be implied by $r=6\sqrt{2}e^{\frac{\pi}{4}i}$	M1 A1	3.1a 1.1b
	Adding multiples of $\frac{2\pi}{5}$ to their argument $z=6\sqrt{2}e^{\frac{\pi}{4}i}\times e^{\frac{2\pi k}{5}i}$ or $z=6\sqrt{2}\left[\cos\left(\frac{\pi}{4}+\frac{2\pi k}{5}\right)+i\sin\left(\frac{\pi}{4}+\frac{2\pi k}{5}\right)\right]$	M1	1.1b
	$z=re^{\left(\theta+\frac{2\pi}{5}\right)i},\ re^{\left(\theta+\frac{4\pi}{5}\right)i},\ re^{\left(\theta+\frac{6\pi}{5}\right)i},\ re^{\left(\theta+\frac{8\pi}{5}\right)i}$ o.e. or $z=re^{\left(\theta+\frac{2\pi}{5}\right)i},\ re^{\left(\theta-\frac{2\pi}{5}\right)i},\ re^{\left(\theta-\frac{6\pi}{5}\right)i},\ re^{\left(\theta-\frac{8\pi}{5}\right)i}$ o.e.	A1ft	1.1b
	$z=6\sqrt{2}e^{\frac{13\pi}{20}i},\ 6\sqrt{2}e^{\frac{21\pi}{20}i},\ 6\sqrt{2}e^{\frac{29\pi}{20}i},\ 6\sqrt{2}e^{\frac{37\pi}{20}i}$ o.e. or $z=6\sqrt{2}e^{\frac{13\pi}{20}i},\ 6\sqrt{2}e^{-\frac{19\pi}{20}i},\ 6\sqrt{2}e^{-\frac{11\pi}{20}i},\ 6\sqrt{2}e^{-\frac{3\pi}{20}i}$ o.e.	A1	1.1b
		(5)	
(ii)(a)	Circle centre (0, 2) and radius 2 or  with the point on the origin	B1	1.1b
	Fully correct 	B1	1.1b
		(2)	
(ii)(b)	$\text{area}=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}4\sin\theta^{\ 2}\ d\theta$ or $\text{area}=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\alpha\sin\theta^{\ 2}\ d\theta$	M1	3.1a
	Uses $\sin^2\theta=\frac{1}{2}-\frac{1}{2}\cos 2\theta$ and integrates to the form $A\theta+B\sin 2\theta$ $\text{area}=8\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\sin^2\theta\ d\theta=4\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}1-\cos 2\theta\ d\theta=4\theta-2\sin 2\theta$	M1	3.1a
	Uses the limits of $\frac{\pi}{4}$ and $\frac{\pi}{3}$ and subtracts the correct way around $\left[4\left(\frac{\pi}{3}\right)-2\sin\left(\frac{2\pi}{3}\right)\right]-\left[4\left(\frac{\pi}{4}\right)-2\sin\left(\frac{2\pi}{4}\right)\right]$	M1	1.1b

	Area = $\frac{\pi}{3}-\sqrt{3}+2$	A1	1.1b
		(4)	
	<u>Alternative</u> 		
	Finds either the areas 1 or 2 Area 1 $=\frac{1}{2}\times 2^2\times\sin\left(\frac{2\pi}{3}\right)\left\{=\sqrt{3}\right\}$ Area 2 $=\frac{1}{2}\times 2^2\times\frac{\pi}{3}\left\{=\frac{2\pi}{3}\right\}$	M1	1.1b
	A complete method to find area 3 Area 3 $=\frac{1}{4}\pi\times 2^2-\frac{1}{2}\times 2^2\left\{=\pi-2\right\}$	M1	3.1a
	A complete method to find the required area Shaded area = Area of semi circle − area 1 − area 2 − area 3 $=\left[\frac{1}{2}\pi\times 2^2\right]-\left[\frac{1}{2}\times 2^2\times\sin\left(\frac{2\pi}{3}\right)\right]-\left[\frac{1}{2}\times 2^2\times\frac{\pi}{3}\right]-\left[\frac{1}{4}\pi\times 2^2-\frac{1}{2}\times 2^2\right]$ $=2\pi-\sqrt{3}-\frac{2\pi}{3}-(\pi-2)$ Or Shaded area = Area of sector − area 1 − area 3 $=\left[\frac{1}{2}\times 4\times\left(\frac{2\pi}{3}\right)\right]-\left[\frac{1}{2}\times 2^2\times\sin\left(\frac{2\pi}{3}\right)\right]-\left[\frac{1}{4}\pi\times 2^2-\frac{1}{2}\times 2^2\right]$ $=\frac{4\pi}{3}-\sqrt{3}-(\pi-2)$	M1	3.1a
	Area = $\frac{\pi}{3}-\sqrt{3}+2$	A1	1.1b
		(4)	
			(11 marks)

Notes:

(i)
M1: Finds the modulus and argument of *z*
A1: Correct modulus and argument of *z*

M1: Uses a correct method to find to all the other 4 vertices of the pentagon. Must be doing the equivalent of adding/ subtracting multiplies of $\frac{2\pi}{5}$ to the argument.

A1ft: All 4 vertices following through on their modulus and argument. Does not need to be simplified for this mark.

A1: All 4 vertices correct in the required form

(ii)(a)

B1: Circle centre (0, 2) and radius 2 or  with the vertex on the origin.

B1: Fully correct region shaded.

(ii) (b)

M1: Writes the required area using polar coordinates

M1: Uses $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ and integrates to the form $A\theta + B\sin 2\theta$

M1: Uses the limits of $\frac{\pi}{4}$ and $\frac{\pi}{3}$ and subtracts the correct way around. Must be some attempt at

$\text{area} = \frac{1}{2}\int \alpha\sin\theta^{\;2}\ \mathrm{d}\theta$ and integration.

A1: Correct exact area = $\frac{\pi}{3} - \sqrt{3} + 2$

Alternative

M1: Finds either area 1 or area 2

M1: A complete method to find the area 3

M1: A complete method to find the required area $=$ Area of semi circle $-$ area 1 $-$ area 2 $-$ area 3 or $=$ Area of sector $-$ area 1 $-$ area 3

A1: Correct exact area = $\frac{\pi}{3} - \sqrt{3} + 2$

9. (a) Given that $|z| < 1$, write down the sum of the infinite series

$$1 + z + z^2 + z^3 + \dots$$

(1)

(b) Given that $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$,

(i) use the answer to part (a), and de Moivre's theorem or otherwise, to prove that

$$\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \dots \quad = \frac{2\sin\theta}{5 - 4\cos\theta}$$

(5)

(ii) show that the sum of the infinite series $1 + z + z^2 + z^3 + \dots$ cannot be purely imaginary, giving a reason for your answer.

(2)

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Question	Scheme	Marks	AOs
9(a)	$\frac{1}{1-z}$	B1	2.2a
		(1)	
(b)(i)	$1+z+z^2+z^3+...$ $=1+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^3+...$ $=1+\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)+\frac{1}{4}(\cos2\theta+\mathrm{i}\sin2\theta)+\frac{1}{8}(\cos3\theta+\mathrm{i}\sin3\theta)+...$	M1	3.1a
	$\frac{1}{1-z}=\frac{1}{1-\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)}\times\frac{1-\frac{1}{2}\cos\theta+\frac{1}{2}\mathrm{i}\sin\theta}{1-\frac{1}{2}\cos\theta+\frac{1}{2}\mathrm{i}\sin\theta}$ or $\frac{1}{1-z}=\frac{2}{2-(\cos\theta+\mathrm{i}\sin\theta)}\times\frac{2-(\cos\theta-\mathrm{i}\sin\theta)}{2-(\cos\theta-\mathrm{i}\sin\theta)}$	M1	3.1a
	$\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin2\theta)+\frac{1}{8}(\sin3\theta)+...\right\}=\frac{\frac{1}{2}\sin\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2}$ or $\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin2\theta)+\frac{1}{8}(\sin3\theta)+...\right\}=\frac{2\sin\theta}{(2-\cos\theta)^2+(\sin\theta)^2}$	M1	2.1
	$\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2=1-\cos\theta+\frac{1}{4}\cos^2\theta+\frac{1}{4}\sin^2\theta$ $=\frac{5}{4}-\cos\theta$ or $(2-\cos\theta)^2+(\sin\theta)^2=4-4\cos\theta+\cos^2\theta+\sin^2\theta$ $=5-4\cos\theta$	M1	1.1b
	$\frac{1}{2}\sin\theta+\frac{1}{4}\sin2\theta+\frac{1}{8}\sin3\theta+...=\frac{\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta}=\frac{2\sin\theta}{5-4\cos\theta}$ *	A1*	1.1b
	Alternative $1+z+z^2+z^3+...$ $=1+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)\right)^3+...$ $=1+\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)+\frac{1}{4}(\cos2\theta+\mathrm{i}\sin2\theta)+\frac{1}{8}(\cos3\theta+\mathrm{i}\sin3\theta)+...$	M1	3.1a

	$\frac{1}{1-z}=\frac{1}{1-\frac{1}{2}e^{i\theta}}\times\frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{2}e^{-i\theta}}$	M1	3.1a
	$\frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{4}e^{i\theta}-\frac{1}{4}e^{-i\theta}+\frac{1}{4}}=\frac{4-2e^{-i\theta}}{5-2\left(e^{i\theta}+e^{-i\theta}\right)}=\frac{4-2(\cos\theta-i\sin\theta)}{5-2(2\cos\theta)}$	M1	2.1
	Select the imaginary part $\frac{2\sin\theta}{5-4\cos\theta}$	M1	1.1b
	$\frac{1}{2}\sin\theta+\frac{1}{4}\sin 2\theta+\frac{1}{8}\sin 3\theta+...=\frac{2\sin\theta}{5-4\cos\theta}$ *	A1*	1.1b
		(5)	
(b)(ii)	$\frac{1-\frac{1}{2}\cos\theta}{\frac{5}{4}-\cos\theta}=0\Rightarrow\cos\theta=2$	M1	3.1a
	As $(-1\leq)\cos\theta\leq 1$ therefore there is no solution to $\cos\theta=2$ so there will also be a real part, hence the sum cannot be purely imaginary.	A1	2.4
	Alternative 1 Imaginary part is $\frac{4-2\cos\theta}{5-4\cos\theta}=\frac{1}{2}+\frac{3}{2(5-4\cos\theta)}$	M1	3.1a
	$-1\leq\cos\theta\leq 1$ therefore $\frac{1}{6}<\frac{3}{2(5-4\cos\theta)}<\frac{3}{2}$ so sum must contain real part	A1	2.4
	Alternative 2 $\frac{1}{1-z}=ki\Rightarrow z=1+\frac{i}{k}$	M1	3.1a
	mod $z>1$ contradiction hence cannot be purely imaginary	A1	2.4
		(2)	
			(8 marks)

Notes:

(a)
B1: See scheme

(b)(i)

M1: Substitutes $z=\frac{1}{2}(\cos\theta+i\sin\theta)$ into at least 3 terms of the series and applies de Moivre's theorem.

M1: Substitutes $z=\frac{1}{2}(\cos\theta+i\sin\theta)$ into their answer to part (a) and rationalises the denominator.

M1: Equates the imaginary terms.
M1: Multiplies out the denominator and simplifies by using the identity $\cos^2\theta+\sin^2\theta=1$

A1*: cso**.** Achieves the printed answer having substituted $z=\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)$ into 4 terms of the series. Alternative **M1:** Substitutes $z=\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)$ into at least 3 terms of the series and applies de Moivre's theorem. **M1:** Substitutes $z=\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}$ into their answer to part (a) and rationalises the denominator. **M1:** Uses $\mathrm{e}^{-\mathrm{i}\theta}=\cos\theta-\mathrm{i}\sin\theta$ and $\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}=2\cos\theta$ to express in terms of $\sin\theta$ and $\cos\theta$ **M1:** Select the imaginary terms. **A1*:** cso Achieves the printed answer having substituted $z=\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)$ into 4 terms of the series.
(b)(ii) **M1:** Setting the real part of the series = 0 and rearranges to find $\cos\theta=...$ **A1:** See scheme
Alternative 1 **M1:** Rearranges imaginary part so that $\cos\theta$ only appears once **A1:** Uses $-1\le\cos\theta\le1$ to show that the sum must always be positive so must contain a real part
Alternative 2 **M1:** Sets sum as purely imaginary and rearranges to make *z* the subject **A1:** Shows a contradiction and draws an appropriate conclusion

1. A student was asked to answer the following:

For the complex numbers $z_1 = 3 - 3\text{i}$ and $z_2 = \sqrt{3} + \text{i}$, find the value of $\arg\left(\dfrac{z_1}{z_2}\right)$

The student's attempt is shown below.

Line 1 → $\arg(z_1) = \tan^{-1}\left(\dfrac{3}{3}\right) = \dfrac{\pi}{4}$

Line 2 → $\arg(z_2) = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

Line 3 → $\arg\left(\dfrac{z_1}{z_2}\right) = \dfrac{\arg(z_1)}{\arg(z_2)}$

Line 4 → $= \left(\dfrac{\pi}{4}\right) \Big/ \left(\dfrac{\pi}{6}\right) = \dfrac{3}{2}$

The student made errors in line 1 and line 3

Correct the error that the student made in

(a) (i) line 1

(ii) line 3

(2)

(b) Write down the correct value of $\arg\left(\dfrac{z_1}{z_2}\right)$

(1)

Question	Scheme	Marks	AOs
1(a) (i) **(a) (ii)**	$\{arg(z_1) =\}\, tan^{-1}\left(\frac{-3}{3}\right)$ or $\{arg(z_1) =\}\, tan^{-1}(-1)$ or $\{arg(z_1) =\} - tan^{-1}\left(\frac{3}{3}\right)$ or $\{arg(z_1) =\} - \frac{\pi}{4}$ or $\{arg(z_1) =\} 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ or states should be – 3 not 3 on top	B1	2.3
	States that $\left\{arg\left(\frac{z_1}{z_2}\right) =\right\} arg(z_1) - arg(z_2)$ Or states that the arguments should be subtracted	B1	2.3
		(2)	
(b)	$\left\{arg\left(\frac{z_1}{z_2}\right) = \left(\text{their} - \frac{\pi}{4}\right) - \frac{\pi}{6} =\right\} - \frac{5\pi}{12}$ Or $\left\{arg\left(\frac{z_1}{z_2}\right) = \left(\text{their}\frac{7\pi}{4}\right) - \frac{\pi}{6}\right\} = \frac{19\pi}{12}$	B1ft	2.2a
		(1)	
		(3 marks)	

Notes:

(a) (i)
B1: See scheme, Condone – 45
Any incorrect arguments seen is B0.
$arg(z_1) = tan^{-1}\left(\frac{3}{-3}\right)$ is B0
Note: They used 3 instead of – 3 is B0, there are two 3's in line 1 do they mean both should – 3
It should be negative is B0
(a) (ii)
B1: See scheme
(b)
B1ft: States a correct value for $arg\left(\frac{z_1}{z_2}\right)$ Follow through on their answer to part (a) (i), do not ISW

4. (i) Given that

$$z_1 = 6e^{\frac{\pi}{3}i} \text{ and } z_2 = 6\sqrt{3}e^{\frac{5\pi}{6}i}$$

show that

$$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$$

(3)

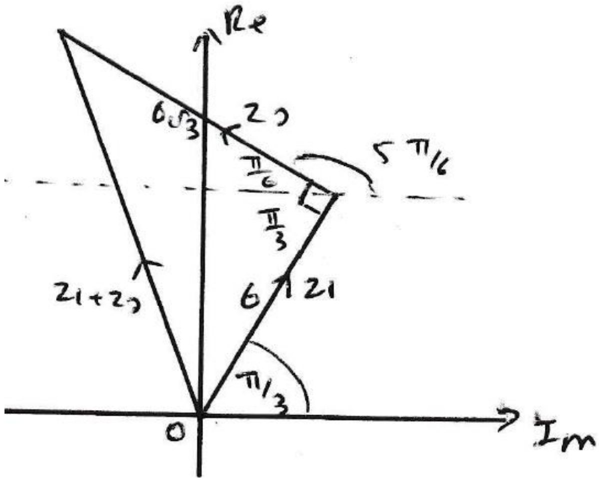
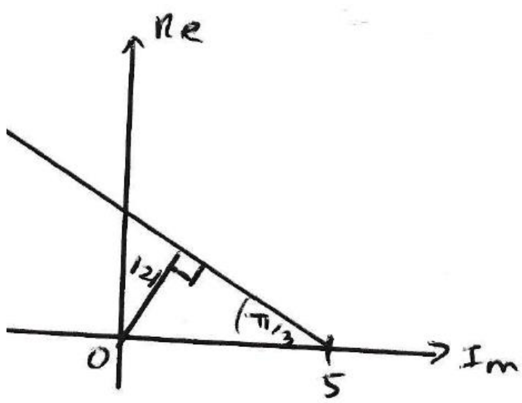
(ii) Given that

$$\arg(z - 5) = \frac{2\pi}{3}$$

determine the least value of $|z|$ as z varies.

(3)

<table>
<tr><th>Question</th><th colspan="2">Scheme</th><th>Marks</th><th>AOs</th></tr>
<tr><td rowspan="4">4(i)</td><td colspan="2">$z_1 = 6\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right] = \ldots\{3 + 3\sqrt{3}i\}$
$z_2 = 6\sqrt{3}\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right] = \ldots\{-9 + 3\sqrt{3}i\}$
$\{z_1 + z_2 =\}(3 + 3\sqrt{3}i) + (-9 + 3\sqrt{3}i) = \ldots\{-6 + 6\sqrt{3}i\}$
Or $\{z_1 + z_2 =\}6\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right] + 6\sqrt{3}\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right] = a + bi$ where a and b are constants, the trig function must be evaluated</td><td>M1</td><td>3.1a</td></tr>
<tr><td>Clearly show the method to find modulus and argument for $z_1 + z_2$
$arg(z_1 + z_2) = \pi - \tan^{-1}\left(\frac{6\sqrt{3}}{6}\right)$
or $\tan^{-1}\left(\frac{6\sqrt{3}}{-6}\right) = \ldots\left\{\frac{2\pi}{3}\right\}$
and
$|z_1 + z_2| = \sqrt{6^2 + (6\sqrt{3})^2} = \ldots\{12\}$</td><td>Alternative 1
$-6 + 6\sqrt{3}i = 12\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$= 12\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$
Alternative 2
$12e^{\frac{2\pi}{3}i} = 12\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$
$= \ldots\{-6 + 6\sqrt{3}i\}$</td><td>dM1</td><td>2.1</td></tr>
<tr><td>$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$ *</td><td>$12e^{\frac{2\pi}{3}i} = -6 + 6\sqrt{3}i$
Therefore $z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$*</td><td>A1*</td><td>1.1b</td></tr>
<tr><td colspan="2"></td><td>(3)</td><td></td></tr>
<tr><td rowspan="4"></td><td colspan="2">Alternative 3
$z_1 + z_2 = 6\boldsymbol{e}^{\frac{\pi}{3}\boldsymbol{i}} + 6\sqrt{3}\boldsymbol{e}^{\frac{5\pi}{6}\boldsymbol{i}}$
$= 12\left[\frac{1}{2}\boldsymbol{\cos}\left(\frac{\pi}{3}\right) + \frac{1}{2}\boldsymbol{i\sin}\left(\frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}\boldsymbol{\cos}\left(\frac{5\pi}{6}\right) + \frac{\sqrt{3}}{2}\boldsymbol{i\sin}\left(\frac{5\pi}{6}\right)\right]$</td><td>M1</td><td>3.1a</td></tr>
<tr><td colspan="2">$12\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 12\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$</td><td>dM1</td><td>2.1</td></tr>
<tr><td colspan="2">$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$*</td><td>A1*</td><td>1.1b</td></tr>
<tr><td colspan="2"></td><td>(3)</td><td></td></tr>
<tr><td rowspan="2"></td><td colspan="2">Alternative 4
$z_1 + z_2 = 6e^{\frac{\pi}{3}i} + 6\sqrt{3}e^{\frac{5\pi}{6}i} = 6e^{\frac{\pi}{3}i}\left(1 + \sqrt{3}e^{\frac{\pi}{2}i}\right) = 6e^{\frac{\pi}{3}i}(1 + \sqrt{3}i)$</td><td>M1</td><td></td></tr>
<tr><td colspan="2">Either $r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$ and $arg = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$</td><td>dM1</td><td></td></tr>
</table>

	Or $6e^{\frac{\pi}{3}i}(1+\sqrt{3}i) = 12e^{\frac{\pi}{3}i}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i()e^{\frac{\pi}{3}i}\left(cos\left(\frac{\pi}{3}\right)+i\,sin\left(\frac{\pi}{3}\right)\right)\right)$		
	$z_1 + z_2 = 12e^{\frac{\pi}{3}i}e^{\frac{\pi}{3}i} = 12e^{\frac{2\pi}{3}i}*$	A1*	
		(3)	
	Alternative 5 Uses geometry to show that $z_1,\ z_2$ and $z_1 + z_2$ form a right-angled triangle 	M1	3.1a
	$arg(z_1+z_2) = \frac{\pi}{3} + tan^{-1}\left(\frac{6\sqrt{3}}{6}\right) = \ldots \left\{\frac{2\pi}{3}\right\}$ $\lvert z_1+z_2\rvert = \sqrt{(6)^2+(6\sqrt{3})^2} = \ldots\{12\}$	dM1	1.1b
	$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}\ *$	A1*	1.1b
		(3)	
(ii)		M1	3.1a
	$sin\left(\frac{\pi}{3}\right) = \frac{\lvert z\rvert}{5} \Rightarrow \lvert z\rvert = \ldots$	M1	1.1b
	$\lvert z\rvert = \frac{5\sqrt{3}}{2}$	A1	1.1b
		(3)	

	Alternative 1 Gradient = $-\tan\left(\frac{\pi}{3}\right)$ $c = 5\tan\left(\frac{\pi}{3}\right)$ leading to $y = -\sqrt{3}x + 5\sqrt{3}$ or $\tan\left(\frac{\pi}{3}\right) = \frac{y}{5-x}$ $\lvert z\rvert^2 = x^2 + y^2 = x^2 + (-\sqrt{3}x + 5\sqrt{3})^2 = 4x^2 - 30x + 75$ $\frac{d\lvert z\rvert^2}{dx} = 8x - 30 = 0 \Rightarrow x = \ldots\{3.75\}$ or $\lvert z\rvert^2 = 4(x - 3.75)^2 + 18.75 \Rightarrow x = \ldots\ldots\{3.75\}$	M1	3.1a
	$\lvert z\rvert = \sqrt{4(\text{their}3.75)^2 - 30(\text{their}3.75) + 75}$	M1	1.1b
	$\lvert z\rvert = \frac{5\sqrt{3}}{2}$	A1	1.1b
		(3)	
	Alternative 2 Gradient = $-\tan\left(\frac{\pi}{3}\right)$ $c = 5\tan\left(\frac{\pi}{3}\right)$ leading to $y = -\sqrt{3}x + 5\sqrt{3}$ Perpendicular line through the origin $y = \frac{1}{\sqrt{3}}x$ and find the point of intersection of the two lines $\left(\frac{15}{4}, \frac{5\sqrt{3}}{4}\right)$	M1	3.1a
	Finds the distance from the origin to their point of intersection $\lvert z\rvert = \sqrt{\left(\text{their }\frac{15}{4}\right)^2 + \left(\text{their }\frac{5\sqrt{3}}{4}\right)^2} = \ldots$	M1	1.1b
	$\lvert z\rvert = \frac{5\sqrt{3}}{2}$	A1	1.1b
		(3)	
			(6 marks)

Notes:

(i)
M1: A complete method to find both $z_1$and z_2 in the form $a + bi$and adds them together.
dM1: Dependent on previous method mark, finds the modulus and argument of $z_1 + z_2$. They must show their method, just stating modulus = 12 and argument = $\frac{2\pi}{3}$ is not sufficient as this is a show question.
Alternative 1: Factorises out 12 and find the argument

Alternative 2: uses $12e^{\frac{2\pi}{3}i} = 12\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \ldots$
A1*: Achieves the correct answer following no errors or omissions.
Alternatively shows that $12e^{\frac{2\pi}{3}i} = -6 + 6\sqrt{3}i$and concludes therefore $z_1 + z_2 = 12e^{\frac{2\pi}{3}i}*$

<u>Alternative 3</u>

M1: Factorises out 12 and writes in the form $12\left[\ldots \boldsymbol{cos}\left(\frac{\pi}{3}\right)+\ldots \boldsymbol{i\,sin}\left(\frac{\pi}{3}\right)+\ldots \boldsymbol{cos}\left(\frac{5\pi}{6}\right)+\ldots \boldsymbol{i\,sin}\left(\frac{5\pi}{6}\right)\right]$

dM1: Dependent on previous mark**.** Writes in the form $12(a+bi)$ leading to the form $12(cos\,\theta + i\,sin\,\theta)$

A1*: Achieves the correct answer following no errors or omissions**.**

Alternative 4

M1: Factorises out 6 and writes in the form $6e^{\frac{\pi}{3}i}\left(1+\sqrt{3}e^{\frac{\pi}{2}i}\right) = 6e^{\frac{\pi}{3}i}(1+ai)$

dM1: Dependent on previous method mark, finds the modulus and argument of $(1+ai)$ or $12(a+bi)$ leading to the form $12(cos\,\theta + i\,sin\,\theta)$

A1*: Achieves the correct answer following no errors or omissions**.**

Alternative 5

M1: Draws a diagram to show that z_1, z_2 and $z_1 + z_2$ form a right-angled triangle.

dM1: Dependent on previous method mark, finds the modulus and argument of $z_1 + z_2$

A1*: Achieves the correct answer following no errors or omissions**.**

Note: Writing $arg(z_1+z_2) = arctan\left(\frac{6\sqrt{3}}{-6}\right) = -\frac{\pi}{3}$ therefore $arg(z_1+z_2) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ with no diagram or finding $z_1 + z_2$ is **M0dM0A0**

(ii)

M1: Draws a diagram and recognises that the shortest distance will form a right-angled triangle.

M1: Uses trigonometry to find the shortest length.

A1: Correct exact value.

Alternative 1

M1: Finds the equation of the half-line by attempting $m = -\,tan\left(\frac{\pi}{3}\right)$ $c = 5\,tan\left(\frac{\pi}{3}\right)$. Finds $x^2 + y^2$ in terms of *x*, differentiates, sets = 0 and finds the value of *x*.

M1: Uses their value of *x* to find the minimum value of $\sqrt{x^2+y^2}$

A1: Correct exact value.

Alternative 2

M1: Finds the equation of the half-line by attempting $m = -\,tan\left(\frac{\pi}{3}\right)$ $c = 5\,tan\left(\frac{\pi}{3}\right)$. Finds the equation of the line perpendicular which passes through the origin. Finds the point of intersection of the lines

M1: Finds the distance from the origin to their point of intersection

A1: Correct exact value.