Cp1Ch8 XMQs and MS

(Total: 97 marks)

1. CP1_Sample Q2 . 6 marks - CP1ch8 Proof by induction
2. CP2_Sample Q3 . 12 marks - CP1ch6 Matrices
3. CP1_Specimen Q7 . 11 marks - CP1ch8 Proof by induction
4. CP1_2019 Q6 . 6 marks - CP1ch8 Proof by induction
5. CP1_2020 Q6 . 12 marks - CP1ch8 Proof by induction
6. CP2_2022 Q3 . 11 marks - CP1ch7 Linear transformations
7. CP(AS)_2018 Q8 . 12 marks - CP1ch8 Proof by induction
8. CP(AS)_2019 Q3 . 6 marks - CP1ch8 Proof by induction
9. CP(AS)_2020 Q8 . 6 marks - CP1ch8 Proof by induction
10. CP(AS)_2021 Q8 . 9 marks - CP1ch8 Proof by induction
11. CP(AS)_2022 Q7 . 6 marks - CP1ch8 Proof by induction

2. Prove by induction that for all positive integers n,

$$f(n) = 2^{3n+1} + 3(5^{2n+1})$$

is divisible by 17

(6)

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Question	Scheme	Marks	AOs
2	When $n = 1$, $2^{3n+1} + 3\left(5^{2n+1}\right) = 16 + 375 = 391$ $391 = 17 \times 23$ so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $2^{3k+1} + 3\left(5^{2k+1}\right)$ is divisible by 17	M1	2.4
	$f(k+1) - f(k) = 2^{3k+4} + 3\left(5^{2k+3}\right) - 2^{3k+1} - 3\left(5^{2k+1}\right)$	M1	2.1
	$= 7 \times 2^{3k+1} + 7 \times 3\left(5^{2k+1}\right) + 17 \times 3\left(5^{2k+1}\right)$		
	$= 7f(k) + 17 \times 3\left(5^{2k+1}\right)$	A1	1.1b
	$f(k+1) = 8f(k) + 17 \times 3\left(5^{2k+1}\right)$	A1	1.1b
	If the statement is true for $n = k$ then it has been shown true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all positive integers n	A1	2.4
		(6)	
			(6 marks)

Notes:

B1: Shows the statement is true for $n = 1$
M1: Assumes the statement is true for $n = k$
M1: Attempts $f(k+1) - f(k)$
A1: Correct expression in terms of $f(k)$
A1: Correct expression in terms of $f(k)$
A1: Obtains a correct expression for $f(k + 1)$
A1: Correct complete conclusion

3. (i)

$$\mathbf{M} = \begin{pmatrix} 2 & a & 4 \\ 1 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

where a is a constant.

(a) For which values of a does the matrix $\mathbf{M}$ have an inverse?

(2)

Given that $\mathbf{M}$ is non-singular,

(b) find $\mathbf{M}^{-1}$ in terms of a

(4)

(ii) Prove by induction that for all positive integers n,

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$$

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Question	Scheme		Marks	AOs
3(i)(a)	$\|\mathbf{M}\| = 2(1+2) - a(-1-1) + 4(2-1) = 0 \Rightarrow a = ...$		M1	2.3
	The matrix **M** has an inverse when $a \neq -5$		A1	1.1b
			(2)	
(b)	$\text{Minors}: \begin{pmatrix} 3 & -2 & 1 \\ -a-8 & 2 & a+4 \\ 4-a & -6 & -2-a \end{pmatrix}$ or $\text{Cofactors}: \begin{pmatrix} 3 & 2 & 1 \\ a+8 & 2 & -a-4 \\ 4-a & 6 & -2-a \end{pmatrix}$		B1	1.1b
	$\mathbf{M}^{-1} = \frac{1}{\|\mathbf{M}\|}\text{adj}(\mathbf{M})$		M1	1.1b
	$\mathbf{M}^{-1} = \frac{1}{2a+10}\begin{pmatrix} 3 & a+8 & 4-a \\ 2 & 2 & 6 \\ 1 & -a-4 & -2-a \end{pmatrix}$	2 correct rows or columns. Follow through their det**M**	A1ft	1.1b
		All correct. Follow through their det**M**	A1ft	1.1b
			(4)	
(ii)	When $n = 1$, lhs $= \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$, rhs $= \begin{pmatrix} 3^1 & 0 \\ 3(3^1-1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ So the statement is true for $n = 1$		B1	2.2a
	Assume true for $n = k$ so $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}$		M1	2.4
	$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$		M1	2.1
	$= \begin{pmatrix} 3 \times 3^k & 0 \\ 3 \times 3(3^k-1)+6 & 1 \end{pmatrix}$		A1	1.1b
	$= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1}-1) & 1 \end{pmatrix}$		A1	1.1b
	If the statement is true for $n = k$ then it has been shown true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all positive integers n		A1	2.4
			(6)	
				(12 marks)

Question 3 notes:	
(i)(a)	
M1:	Attempts determinant, equates to zero and attempts to solve for a in order to establish the restriction for a
A1:	Provides the correct condition for a if **M** has an inverse
(i)(b)	
B1:	A correct matrix of minors or cofactors
M1:	For a complete method for the inverse
A1ft:	Two correct rows following through their determinant
A1ft:	Fully correct inverse following through their determinant
(ii)	
B1:	Shows the statement is true for $n = 1$
M1:	Assumes the statement is true for $n = k$
M1:	Attempts to multiply the correct matrices
A1:	Correct matrix in terms of k
A1:	Correct matrix in terms of $k + 1$
A1:	Correct complete conclusion

7. (i) Prove by induction that, for $n \in \mathbb{N}$,

$$\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^n = \begin{pmatrix} 3n+1 & -n \\ 9n & 1-3n \end{pmatrix}$$

(6)

(ii) Consider the statement

$$n^2 < 2^n \qquad \text{for all } n \in \mathbb{Z}^+$$

A student attempts to prove this statement using induction as follows.

Student's response

For $n = 1$ we have $1^2 = 1$ and $2^1 = 2$
Since $1 < 2$ the statement is true for $n = 1$

Suppose it is true for $n = k$, so $k^2 < 2^k$

Line 4 ⟶ Then $(k+1)^2 = k^2 + 2k + 1 < k^2 + k^2$ (since $2k + 1 < k^2$ for $k \in \mathbb{Z}^+$)
$= 2k^2$
$< 2 \times 2^k$ (by the assumption $k^2 < 2^k$)
$= 2^{k+1}$
Hence the result is true for $n = k + 1$

So the result is true for $n = 1$ and if it is true for $n = k$ then it is true for $n = k + 1$, and hence it is true for all positive integers n by mathematical induction.

(a) Show by a counterexample that the statement is not true.

Given that the only mathematical error in the student's proof occurs in line 4,

(b) identify the error made in the student's proof,

(c) hence determine for which positive integers the statement is true, explaining your reasoning.

(5)

Question	Scheme	Marks	AOs
7(i)	When $n = 1$, LHS = $\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}$, RHS = $\begin{pmatrix} 3\times1+1 & -1 \\ 9\times1 & 1-3\times1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}$. So the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$, so $\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^k = \begin{pmatrix} 3k+1 & -k \\ 9k & 1-3k \end{pmatrix}$	M1	2.4
	Then $\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^{k+1} = \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}\begin{pmatrix} 3k+1 & -k \\ 9k & 1-3k \end{pmatrix}$ or $\begin{pmatrix} 3k+1 & -k \\ 9k & 1-3k \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}$	M1	2.1
	$= \begin{pmatrix} 4(3k+1)-9k & -4k-(1-3k) \\ 9(3k+1)-18k & -9k-2(1-3k) \end{pmatrix}$ or $\begin{pmatrix} 4(3k+1)-9k & -(3k+1)+2k \\ 36k+9(1-3k) & -9k-2(1-3k) \end{pmatrix}$	A1	1.1b
	$= \begin{pmatrix} 3(k+1)+1 & -(k+1) \\ 9(k+1) & 1-3(k+1) \end{pmatrix}$	A1	1.1b
	Hence the result is true for $n = k+1$. Since it is true for $n = 1$, and if true for $k = n$ then true for $n = k+1$, thus by mathematical induction the result holds for all $n \in \mathbb{Z}^+$	A1 cso	2.4
		(6)	
(ii)	(a) $2^2 = 4 \not< 4 = 2^2$ OR $3^2 = 9 \not< 8 = 2^3$ OR $4^2 = 16 \not< 16 = 2^4$	B1	1.1b
	(b) The statement $2k+1 < k^2$ is not true for all positive integers.	B1	1.1b
	(c) The statement in line 4 is true for positive integers $k > 2$ so the induction hypothesis is true for $n > 2$. So the induction holds from any base case greater than 2.	M1	2.3
	Since the result is true for $n = 5$ as $5^2 = 25 < 32 = 2^5$ and $2k+1 < k^2$ also true for $k > 5$ so the induction holds with base case $n = 5$.	A1	2.4
	But not true for $n = 2$, 3 or 4 as $2^2 = 4 \not< 4 = 2^2$ and $3^2 = 9 \not< 8 = 2^3$ and $4^2 = 16 \not< 16 = 2^4$. Hence true for $n = 1$ and for $n \geqslant 5$	A1	2.1
		(5)	
		(11 marks)	

Notes:

(a)

B1: Shows the general form holds for $n = 1$.

M1: Makes the inductive assumption, assume true for $n = k$.

M1: Attempts the multiplication either way.

A1: Correct matrix in terms of k.

A1: Rearranged into correct form to show true for $k + 1$.

A1: Completes the inductive argument conveying **all** three underlined points or equivalent at some point in their argument.

(b)(i)

B1: Provides a suitable counter example using $n = 2$, 3 or 4. Accept = in place of $\nless$ as long as there is a suitable conclusion with it.

(b)(ii)

B1: Identifies the error as in the scheme or equivalent (e.g. $k^2 + 2k + 1 < 2k^2$ is not always true).

(b)(iii)

M1: Identifies that the induction is valid as long as $2k+1 < k^2$ is true which happens for $k \geqslant 3$ (accept any value greater than 3 for this mark).

A1: Correct base case of 5 and explains the proof given holds for integers greater than or equal to 5.

A1: Complete argument correct. All positive integers satisfying the inequality identified, with demonstration that 2, 3 and 4 do not.

6. Prove by induction that for all positive integers n

$$f(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5

(6)

A1*: Puts all the components together to form the given differential equation cso (b) M1: Uses the model to find the integrating factor and attempts the solution of the differential equation. Look for $I.F. = e^{\int \frac{2}{100+t}\,dt} \Rightarrow S \times \text{'their } I.F.\text{'} = \int 3 \times \text{'their } I.F.\text{'}\,dt$ A1: Correct solution condone missing + *c* **For the next three mark there must be a constant of integration** M1: Interprets the initial conditions, $t = 0 \;\; S = 0,$ and uses in their equation to find the constant of integration. dM1: Dependent on having a constant of integration. Uses their solution to the problem to find the amount of salt after 10 minutes. A1: Awrt 27 or $\frac{3310}{121}$. (If the units are stated they must be correct) **Note:** If achieves $S(100+t)^2 = 30\,000t + 300t^2 + t^3 + c$ the constant of integration *c* = 0 and the correct amount of salt can be achieved. If there is no + *c* the maximum they can score is M1A1M0M0A0
Notes continued
(c) **Note:** Look out for setting *S* = 0.9 in this part, which scores no marks. M1: Uses their solution to the model and divides by 100 + *t* as an interpretation of the concentration and sets = 0.9. Alternatively recognises that the amount of salt = 0.9(100 + *t*) and substitutes for *S* in their solution to the model. dM1: Dependent on previous method mark. Solves their equation to obtain a value for *t*. May use a calculator. A1: Awrt 115 (If the units are stated they must be correct) or 1hr 45 mins with units (d) B1: Evaluates the model by making a suitable comment – see scheme for examples.

Question	Scheme	Marks	AOs
6	**Way 1** f(*k* +1) – f(*k*)		
	When *n* = 1, $3^{2n+4} - 2^{2n} = 729 - 4 = 725$ $(725 = 145 \times 5)$ so the statement is true for *n* = 1	B1	2.2a
	Assume true for *n* = *k* so $3^{2k+4} - 2^{2k}$ is divisible by 5	M1	2.4
	$f(k+1) - f(k) = 3^{2k+6} - 2^{2k+2} - 3^{2k+4} + 2^{2k}$	M1	2.1
	either $8f(k) + 5 \times 2^{2k}$ or $3f(k) + 5 \times 3^{2k+4}$	A1	1.1b
	$f(k+1) = 9f(k) + 5 \times 2^{2k}$ or $f(k+1) = 4f(k) + 5 \times 3^{2k+4}$ o.e.	A1	1.1b
	If true for *n* = *k* then it is true for	A1	2.4

	<u>$n = k + 1$</u> and as it is <u>true for $n = 1$</u>, the statement is <u>true for all (positive integers) n</u>. (Allow 'for all values')		
		(6)	
	<u>**Way 2**</u> f(k +1)		
	When $n = 1$, $3^{2n+4} - 2^{2n} = 729 - 4 = 725$ ($725 = 145 \times 5$) so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $3^{2k+4} - 2^{2k}$ is divisible by 5	M1	2.4
	$\mathrm{f}\left(k+1\right) = 3^{2(k+1)+4} - 2^{2(k+1)} \left(= 3^{2k+6} - 2^{2k+2}\right)$	M1	2.1
	$\mathrm{f}\ k+1\ = 9\mathrm{f}\ \ k\ + 5\times 2^{2k}$ or $\mathrm{f}\ \ k+1\ = 4\mathrm{f}\ \ k\ + 5\times 3^{2k+4}$ o.e.	A1 A1	1.1b 1.1b
	<u>If true for $n = k$ then it is true for</u> <u>$n = k + 1$</u> and as it is <u>true for $n = 1$</u>, the statement is <u>true for all (positive integers) n</u>. (Allow 'for all values')	A1	2.4
		(6)	
	<u>**Way 3**</u> f(k) = 5M		
	When $n = 1$, $3^{2n+4} - 2^{2n} = 729 - 4 = 725$ ($725 = 145 \times 5$) so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $3^{2k+4} - 2^{2k} = 5M$	M1	2.4
	$\mathrm{f}\left(k+1\right) = 3^{2(k+1)+4} - 2^{2(k+1)} \left(= 3^{2k+6} - 2^{2k+2}\right)$	M1	2.1
	$\left(\mathrm{f}\left(k+1\right) = 3^2 \times 3^{2k+4} - 2^2 \times 2^{2k} = 3^2 \times \left(5M + 2^{2k+2}\right) - 2^2 \times 2^{2k}\right)$ $\mathrm{f}\ \ k+1\ = 45M + 5\times 2^{2k}$ o.e. OR $\left(\mathrm{f}\left(k+1\right) = 3^2 \times 3^{2k+4} - 2^2 \times 2^{2k} = 3^2 \times 3^{2k+4} - 2^2 \times \left(3^{2k+4} - 5M\right)\right)$ $\mathrm{f}\ \ k+1\ = 5\times 3^{2k+4} + 20M$ o.e.	A1 A1	1.1b 1.1b
	<u>If true for $n = k$ then it is true for</u> <u>$n = k + 1$</u> and as it is <u>true for $n = 1$</u>, the statement is <u>true for all (positive integers) n</u>. (Allow 'for all values')	A1	2.4
		(6)	
	<u>**Way 4**</u> f(k +1) + f(k)		
	When $n = 1$, $3^{2n+4} - 2^{2n} = 729 - 4 = 725$ ($725 = 145 \times 5$) so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $3^{2k+4} - 2^{2k}$ is divisible by 5	M1	2.4
	$\mathrm{f}\left(k+1\right) + \mathrm{f}\left(k\right) = 3^{2k+6} - 2^{2k+2} + 3^{2k+4} - 2^{2k}$	M1	2.1
	$\mathrm{f}\left(k+1\right) + \mathrm{f}\left(k\right) = 3^2 \times 3^{2k+4} - 2^2 \times 2^{2k} + 3^{2k+4} - 2^{2k}$	A1	1.1b

	leading to $10\times3^{2k+4}-5\times2^{2k}$		
	$\text{f}\;\;k+1\;=5\left[2\times3^{2k+4}-2^{2k}\right]-\text{f}\,(k)$ o.e.	A1	1.1b
	<u>If true for *n* = *k* then it is true for</u> <u>*n* = *k* + 1</u> and as it is <u>true for *n* = 1</u>, the statement is <u>true for all</u> <u>(positive integers) *n*</u>. (Allow ‘for all values’)	A1	2.4
		(6)	
	<u>**Way 5**</u> f(*k* +1) – ‘M’f(*k*) (Selecting a value of M that will lead to multiples of 5)		
	When *n* = 1, $3^{2n+4}-2^{2n}=729-4=725$ ($725=145\times5$)so the statement is true for *n* = 1	B1	2.2a
	Assume true for *n* = *k* so $3^{2k+4}-2^{2k}$ is divisible by 5	M1	2.4
	$\text{f}\left(k+1\right)-\text{'M'f}\left(k\right)=3^{2k+6}-2^{2k+2}-\text{'M'}\times3^{2k+4}+\text{'M'}\times2^{2k}$	M1	2.1
	$\text{f}\;\;k+1\;-\text{'M'f}\;\;k\;\;=\;\;9-\text{'M'}\;\;\times3^{2k+4}-\;\;4-\text{'M'}\;\;\times2^{2k}$	A1	1.1b
	$\text{f}\;\;k+1\;=\;\;9-\text{'M'}\;\;\times3^{2k+4}-\;\;4-\text{'M'}\;\;\times2^{2k}+\text{'M'f}\;\;k$ o.e.	A1	1.1b
	<u>If true for *n* = *k* then it is true for</u> <u>*n* = *k* + 1</u> and as it is <u>true for *n* = 1</u>, the statement is <u>true for all</u> <u>(positive integers) *n*</u>. (Allow ‘for all values’)	A1	2.4
		(6)	
			(6 marks)

Notes

<u>**Way 1**</u> f(*k* +1) – f(*k*)

B1: Shows the statement is true for *n* = 1. Needs to show f(1) = 725 and conclusion true for *n* = 1, this statement can be recovered in their conclusion if says e.g. true for *n* = 1

M1: Makes an assumption statement that assumes the result is true for *n* = *k*. Assume (true for) $n=k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n=k$ then …etc

M1: Attempts f(*k* +1) – f(*k*) or equivalent work

A1: Achieves a correct simplified expression for f(*k* +1) – f(*k*)

A1: Achieves a correct expression for f(*k* +1) in terms of f(*k*)

A1: Correct complete conclusion. This mark is dependent on **all** previous marks. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

<u>**Way 2**</u> f(*k* +1)

B1: Shows the statement is true for *n* = 1. Needs to show f(1) = 725 and conclusion true for *n* = 1, this statement can be recovered in their conclusion if says e.g. true for *n* = 1.

M1: Makes an assumption statement that assumes the result is true for *n* = *k*. Assume (true for) $n=k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n=k$ then …etc

M1: Attempts f(*k* +1)

A1: Correctly achieves either $9\text{f}\;\;k$ **or** 5×2^{2k} or either $4\text{f}\;\;k$ **or** $5\times3^{2k+4}$

A1: Achieves a correct expression for f(*k* +1) in terms of f(*k*)
A1: Correct complete conclusion. This mark is dependent on **all** previous marks. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

Way 3 f(*k*) = 5*M*
B1: Shows the statement is true for *n* = 1. Needs to show f(1) = 725 and conclusion true for *n* = 1, this statement can be recovered in their conclusion if says e.g. true for *n* = 1.
M1: Makes an assumption statement that assumes the result is true for *n* = *k*. Assume (true for) $n = k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n = k$ then …etc
M1: Attempts f(*k* +1)
A1: Correctly achieves either $45M$ **or** 5×2^{2k} or either $20M$ **or** $5 \times 3^{2k+4}$
A1: Achieves a correct expression for f(*k* +1) in terms of *M* and 2^{2k} or *M* and 3^{2k+4}
A1: Correct complete conclusion. This mark is dependent on **all** previous marks. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

Way 4 f(*k* +1) + f(*k*)
B1: Shows the statement is true for *n* = 1. Needs to show f(1) = 725 and conclusion true for *n* = 1, this statement can be recovered in their conclusion if says e.g. true for *n* = 1
M1: Makes an assumption statement that assumes the result is true for *n* = *k*. Assume (true for) $n = k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n = k$ then …etc
M1: Attempts f(*k* +1) + f(*k*) or equivalent work
A1: Achieves a correct simplified expression for f(*k* +1) + f(*k*)
A1: Achieves a correct expression for $\mathrm{f}(k+1) = 5\left[2 \times 3^{2k+4} - 2^{2k}\right] - \mathrm{f}(k)$
A1: Correct complete conclusion. This mark is dependent on **all** previous marks. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

Notes continued

Way 5 f(*k* +1) – Mf(*k*) (Selects a suitable value for M which leads to divisibility of 5)
B1: Shows the statement is true for *n* = 1. Needs to show f(1) = 725 and conclusion true for *n* = 1, this statement can be recovered in their conclusion if says e.g. true for *n* = 1
M1: Makes an assumption statement that assumes the result is true for *n* = *k*. Assume (true for) $n = k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n = k$ then …etc
M1: Attempts f(*k* +1) – Mf(*k*) or equivalent work
A1: Achieves a correct simplified expression, $\mathrm{f}(k+1) - \text{'M'}\mathrm{f}(k)$ which is divisible by 5

$$\mathrm{f}(k+1) - \text{'M'}\mathrm{f}(k) = (9 - \text{'M'}) \times 3^{2k+4} - (4 - \text{'M'}) \times 2^{2k}$$

A1: Achieves a correct expression for $\mathrm{f}(k+1) = (9 - \text{'M'}) \times 3^{2k+4} - (4 - \text{'M'}) \times 2^{2k} + \text{'M'}\mathrm{f}(k)$ which is divisible by 5
A1: Correct complete conclusion. This mark is dependent on **all** previous marks. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

6. (i) Prove by induction that for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} (3r + 1)(r + 2) = n(n + 2)(n + 3)$$

(6)

(ii) Prove by induction that for all positive **odd** integers n

$$f(n) = 4^n + 5^n + 6^n$$

is divisible by 15

(6)

Question	Scheme	Marks	AOs
6(i)	When n = 1, $\sum_{r=1}^{1}(3r+1)(r+2)=4\times3=12$ $1(1+2)(1+3)=12$ (so the statement is true for n = 1)	B1	2.2a
	Assume true for $n = k$ so $\sum_{r=1}^{k}(3r+1)(r+2)=k(k+2)(k+3)$	M1	2.4
	$\sum_{r=1}^{k+1}(3r+1)(r+2)=k(k+2)(k+3)+(3k+4)(k+3)$	M1	2.1
	$=(k+3)(k^2+5k+4)$	A1	1.1b
	$\sum_{r=1}^{k+1}(3r+1)(r+2)=(k+1)(k+3)(k+4)$	A1	1.1b
	$\sum_{r=1}^{k+1}(3r+1)(r+2)=(\underline{k+1})(\underline{k+1}+2)(\underline{k+1}+3)$ If the statement is <u>true for $n = k$ then</u> it has been shown <u>true for $n = k + 1$</u> and as it is <u>true for n = 1</u>, the statement <u>is true for all (positive integers) n.</u>	A1	2.4
		(6)	
(ii) Way 1	When n = 1, $4^1+5^1+6^1=15$ so the statement is true for n = 1	B1	2.2a
	Assume true for $n = k$ so $4^k+5^k+6^k$ is divisible by 15	M1	2.4
	$\text{f}(k+2)=4^{k+2}+5^{k+2}+6^{k+2}$	M1	2.1
	$=16\times4^k+16\times5^k+16\times6^k+9\times5^k+20\times6^k$ $=16\text{f}(k)+45\times5^{k-1}+120\times6^{k-1}$	A1 A1	1.1b 1.1b
	E.g As 15 divides f(k), 45 and 120, so 15 divides f(k+1). <u>If true for $n = k$</u> then <u>true for $n = k + 2$</u>, <u>true for n = 1</u> so <u>true for all positive odd integers n</u>	A1	2.4
		(6)	
(ii) Way 2	When n = 1, $4^1+5^1+6^1=15$ so the statement is true for n = 1	B1	2.2a
	Assume true for $n = k$ so $4^k+5^k+6^k$ is divisible by 15	M1	2.4
	$\text{f}(k+2)-\text{f}(k)=4^{k+2}+5^{k+2}+6^{k+2}-4^k-5^k-6^k$	M1	2.1
	$=15\times4^k+24\times5^k+35\times6^k$ $=15\text{f}(k)+45\times5^{k-1}+120\times6^{k-1}$	A1	1.1b
	$\text{f}(k+2)=16\text{f}(k)+45\times5^{k-1}+120\times6^{k-1}$	A1	1.1b
	E.g $\text{f}(k+2)=16\text{f}(k)+15(3\times5^{k-1}+8\times6^{k-1})$ so <u>if true for $n = k$</u> then <u>true for $n = k + 2$</u>, <u>true for n = 1</u> so <u>true for all positive odd integers n</u>	A1	2.4
		(6)	

(ii) Way 3	When $n = 1$, $4^1 + 5^1 + 6^1 = 15$ so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = 2k+1$ so f(2k+1) = $4^{2k+1} + 5^{2k+1} + 6^{2k+1}$ is divisible by 15	M1	2.4
	$\text{f}\left(2k+3\right) = 4^{2k+3} + 5^{2k+3} + 6^{2k+3}$ or $\text{f}\left(2k+3\right) - \text{f}\left(2k+1\right) = 4^{2k+3} + 5^{2k+3} + 6^{2k+3} - 4^{2k+1} - 5^{2k+1} - 6^{2k+1}$	M1	2.1
	$\text{f}(2k+3) = 16\times4^{2k+1} + 25\times5^{2k+1} + 36\times6^{2k+1}$ $= 16\left(4^{2k+1} + 5^{2k+1} + 6^{2k+1}\right) + 9\times5^{2k+1} + 20\times6^{2k+1}$ OR $\text{f}(2k+3) - \text{f}(2k+1) = 16\times4^{2k+1} + 25\times5^{2k+1} + 36\times6^{2k+1} - 4^{2k+1} - 5^{2k+1} - 6^{2k+1}$ $= 15\times4^{2k+1} + 120\times5^{2k} + 210\times6^{2k}$	A1	1.1b
	$\text{f}(2k+3) = 16\left(4^{2k+1} + 5^{2k+1} + 6^{2k+1}\right) + 45\times5^{2k} + 120\times6^{2k}$ OR $\text{f}\left(2k+3\right) = \text{f}\left(2k+1\right) + 15\times4^{2k+1} + 120\times5^{2k} + 210\times6^{2k}$	A1	1.1b
	$\text{f}(2k+3) = 16\left(4^{2k+1} + 5^{2k+1} + 6^{2k+1}\right) + 15\left(3\times5^{2k} + 8\times6^{2k}\right)$ OR $\text{f}\left(2k+3\right) = \text{f}\left(2k+1\right) + 15\left(4^{2k+1} + 8\times5^{2k} + 14\times6^{2k}\right)$ and If true for $n = 2k+1$ then true for $n = 2k+3$, true for $n = 1$ so true for all positive odd integers n	A1	2.4
		(6)	
			(12 marks)

Notes

(i)
B1: Shows the statement is true for $n = 1$ by evaluating **both** sides. There is no need for statement "hence true for n =1" for this mark but if they never state this the final A will be forfeited.
Look for a minimum of $4\times3=12$ for the LHS and $1\times3\times4$ for the RHS. If only one side is explicitly evaluated, it is B0, but all other marks may be gained.
M1: Makes an assumption statement that assumes the result is true for $n = k$
M1: Makes the inductive step by attempting to add the $(k + 1)^{\text{th}}$ term to the assumed result. Attempts at using the standard summation formulae score M0, as the question requires induction.
A1: Correct expression with at least one correct linear factor
A1: Obtains a fully correct factorised expression. May be as in scheme or in terms of k + 1.
A1: Correct complete conclusion with all ideas conveyed at the end or as a narrative and the sum to k+1 expressed in terms of k + 1 (or with the expression in term of k + 1 stated earlier – it must be seen at some stage). Allow slips in notation if the intent is correct. Depends on all except the B mark, though an attempt at checking the n = 1 case must have been made.
(ii) **Way 1**
B1: Shows that f(1) = 15
M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.)
M1: Attempts f(k + 2)

A1: Correctly obtains 16f(k) **or** $45\times5^{k-1}+120\times6^{k-1}$
A1: Reaches a correct expression for f(k + 2) in terms of f(k)
A1: Correct conclusion, including explanation that all terms on RHS are divisible by 15, or writing terms each explicitly as multiples of 15. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution. Do not allow "true for all n" where n represents natural numbers, as this is incorrect.
Way 2
B1: Shows that f(1) = 15
M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.)
M1: Attempts f(k + 2) – f(k) or equivalent work
A1: Achieves a correct expression for f(k + 2) – f(k) in terms of f(k)
A1: Reaches a correct expression for f(k + 2) in terms of f(k)
A1: Correct conclusion, including explanation that all terms on RHS are divisible by 15, or writing terms each explicitly as multiples of 15. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution. Do not allow "true for all n" where n represents natural numbers, as this is incorrect.

Way 3
B1: Shows that f(1) = 15
M1: Makes a statement that assumes the result is true for some odd value of n (Assume (true for) $n = 2k + 1$ is sufficient (or may use $2k-1$ – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.)
M1: Attempts f(2k+3) (like Way 1), or f(2k +3) – f(2k+1) or equivalent work (like Way 2) It must be the correct f(2k+1) used, in the latter case (subtracting $4^k+5^k+6^k$ instead of $4^{2k+1}+5^{2k+1}+6^{2k+1}$ is M0)
A1: Reaches $16\left(4^{2k+1}+5^{2k+1}+6^{2k+1}\right)+9\times5^{2k+1}+20\times6^{2k+1}$ or suitable equivalent OR achieves a correct expression for f(2k +3) – f(2k+1) in terms of f(2k+1) where factors of 15 are apparent.
A1: Reaches $16\left(4^{2k+1}+5^{2k+1}+6^{2k+1}\right)+45\times5^{2k}+120\times6^{2k}$ or a suitable equivalent OR a correct expression for f(2k + 3) of form $A\text{f}(2k+1)+15(...)$ (though the 15(…) could be separate multiples of 15 where the 15 need not yet be extracted).
A1: Correct conclusion, including explanation that all terms on RHS are divisible by 15, or writing terms each explicitly as multiples of 15. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution. Do not allow "true for all n" where n represents natural numbers, as this is incorrect, but if n represents odd numbers than allow.
Note: $4^{2k}=16^k$ etc may be used throughout.
Note: Way 3 should apply to cases where the question is rephrased in terms of $4^{2n+1}+5^{2n+1}+6^{2n+1}$ for all n (rather than odd n).
Accept use of alternative equivalent language throughout.

3. $\mathbf{M} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$ where a is a constant

(a) Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\mathbf{M}^n = \begin{pmatrix} 3^n & \frac{a}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix}$$

(6)

Triangle T has vertices A, B and C.

Triangle T is transformed to triangle T' by the transformation represented by $\mathbf{M}^n$ where $n \in \mathbb{N}$

Given that

- triangle T has an area of $5\,\text{cm}^2$
- triangle T' has an area of $1215\,\text{cm}^2$
- vertex $A(2, -2)$ is transformed to vertex $A'(123, -2)$

(b) determine

(i) the value of n

(ii) the value of a

(5)

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Question	Scheme	Marks	AOs
3(a)	$n=1 \Rightarrow \mathbf{M}^1 = \begin{pmatrix} 3^1 & \frac{a}{2}(3^1-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$ {So the result is true for $n=1$}	B1	2.2a
	Assume true for $n=k$ Or assume $\mathbf{M}^n$ or $\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & \frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix}$	M1	2.4
	A correct method to find an expression for $n=k+1$ $\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3^k & \frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3^k & \frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix}$	M1	1.1b
	$\begin{pmatrix} 3(3^k) & a(3^k)+\frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 3(3^k) & 3\times\frac{a}{2}(3^k-1)+a \\ 0 & 1 \end{pmatrix}$	A1	1.1b
	$\begin{pmatrix} 3^{k+1} & \frac{a}{2}[2(3^k)+(3^k-1)] \\ 0 & 1 \end{pmatrix} =$ $\begin{pmatrix} 3^{k+1} & \frac{a}{2}[3(3^k)-1] \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3^{k+1} & \frac{a}{2}[3^{k+1}-1] \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 3(3^k) & 3\times\frac{a}{2}(3^k-1)+a \\ 0 & 1 \end{pmatrix} =$ $\begin{pmatrix} 3^{k+1} & \frac{a}{2}(3(3^k-1)+2) \\ 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 3^{k+1} & \frac{a}{2}(3^{k+1}-1) \\ 0 & 1 \end{pmatrix}$	A1	2.1
	If true for $n=k$ **then** true for $n=k+1$ and as it is **true** for $n=1$ the statement is **true for** all (positive integers) ***n***	A1	2.4
		(6)	
(b)(i)	$det(\mathbf{M}^n) = 3^n$ or $det(\mathbf{M}) = 3$	B1	1.1b
	Uses $5 \times det(\mathbf{M}^n) = 1215 \Rightarrow p^n = q \Rightarrow n = \ldots$ $5 \times 3^n = 1215 \Rightarrow 3^n = 243 \Rightarrow n = \ldots$	M1	3.1a
	$n=5$	A1	1.1b
(ii)	$\begin{pmatrix} 3^n & \frac{a}{2}(3^n-1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 123 \\ -2 \end{pmatrix} \Rightarrow 2(3^n) - 2\frac{a}{2}(3^n-1) = 123$ $\Rightarrow a = \ldots$	M1	1.1b

	$\begin{pmatrix} 243 & \frac{a}{2}(243-1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 123 \\ -2 \end{pmatrix} \Rightarrow 2(243) - 2\frac{a}{2}(243-1) = 123 \Rightarrow a = \ldots$ $\frac{1}{243}\begin{pmatrix} 1 & -\frac{a}{2}(243-1) \\ 0 & a \end{pmatrix}\begin{pmatrix} 123 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \Rightarrow \frac{123 - 2\frac{a}{2}(243-1)}{243} = -2 \Rightarrow a = \ldots$		
	$a = 1.5$	A1	1.1b
		(5)	
			(11 marks)

Notes:

(a)

B1: Shows that the result holds for $n = 1$. Must see substitution in the RHS minimum required is $\begin{pmatrix} 3 & \frac{a}{2}(3-1) \\ 0 & 1 \end{pmatrix}$ and reaches $\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$

M1: Assumes the result is true for some value of $n = k$. Assume (true for) $n = k$ is sufficient.

Alternatively states assume $\mathbf{M}^n$ or $\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & \frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix}$

M1: Sets up a matrix multiplication of their assumed result multiplied by the original matrix, either way round. Allow a slip as long as the intention is clear.

A1: Achieves a correct un-simplified matrix

A1: Reaches a correct simplified matrix with **no errors, the correct un-simplified matrix seen previously and at least one intermediate line which must be correct.**

A1: Correct conclusion. This mark is dependent on all previous marks except B mark but n = 1 must have been attempted. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution. Condone $n \in \mathbb{Z}$

(b)(i)

B1: States correct determinant. This can be implied by a correct equation

M1: Correct method to find a value of n using $5 \times$ 'their $det(\mathbf{M}^n)$' $= 1215$ which involves solving an index equation of the form $p^n = q$ where $n > 1$

A1: $n = 5$

(ii)

M1: Sets up an equation by multiplying the matrix $\mathbf{M}^n$ by $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ setting equal to $\begin{pmatrix} 123 \\ -2 \end{pmatrix}$ and reaches a value for a. You may just see $2(3^n) - 2\frac{a}{2}(3^n - 1) = 123 \Rightarrow a = \ldots$

Follow through on their value for n.

A1: $a = 1.5$

8. (i) Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 4n+1 & -8n \\ 2n & 1-4n \end{pmatrix}$$

(6)

(ii) Prove by induction that for $n \in \mathbb{Z}^+$

$$f(n) = 4^{n+1} + 5^{2n-1}$$

is divisible by 21

(6)

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Question	Scheme	Marks	AOs
8(i)	$n=1, \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^1 = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}, \quad \begin{pmatrix} 4\times1+1 & -8(1) \\ 2\times1 & 1-4(1) \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$ So the result is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k = \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}$	M1	2.4
	$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^{k+1} = \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$ or $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^{k+1} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}$	M1	1.1b
	$\begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 5(4k+1)-16k & -8(4k+1)+24k \\ 10k+2(1-4k) & -16k-3(1-4k) \end{pmatrix}$ or $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix} = \begin{pmatrix} 5(4k+1)-16k & -40k-8(1-4k) \\ 2(1+4k)-6k & -16k-3(1-4k) \end{pmatrix}$	A1	1.1b
	$= \begin{pmatrix} 4(k+1)+1 & -8(k+1) \\ 2(k+1) & 1-4(k+1) \end{pmatrix}$	A1	2.1
	If true for $n = k$ then true for $n = k + 1$, true for $n = 1$ so true for all (positive integers) n (Allow "for all values")	A1	2.4
		(6)	
(ii) **Way 1**	$f(k + 1) - f(k)$		
	When $n = 1$, $4^{n+1} + 5^{2n-1} = 16 + 5 = 21$ so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $4^{k+1} + 5^{2k-1}$ is divisible by 21	M1	2.4
	$f(k+1) - f(k) = 4^{k+2} + 5^{2k+1} - 4^{k+1} - 5^{2k-1}$	M1	2.1
	$= 4\times4^{k+1} + 25\times5^{2k-1} - 4^{k+1} - 5^{2k-1}$		
	$= 3f(k) + 21\times5^{2k-1}$ or e.g. $= 24f(k) - 21\times4^{k+1}$	A1	1.1b
	$f(k+1) = 4f(k) + 21\times5^{2k-1}$ or e.g. $f(k+1) = 25f(k) - 21\times4^{k+1}$	A1	1.1b
	If true for $n = k$ then true for $n = k + 1$, true for $n = 1$ so true for all (positive integers) n (Allow "for all values")	A1	2.4
		(6)	

(ii) **Way 2**	f(k + 1)		
	When n = 1, $4^{n+1}+5^{2n-1}=16+5=21$ so the statement is true for n = 1	B1	2.2a
	Assume true for n = k so $4^{k+1}+5^{2k-1}$ is divisible by 21	M1	2.4
	$\text{f}\left(k+1\right)=4^{k+1+1}+5^{2(k+1)-1}$	M1	2.1
	$\text{f}\left(k+1\right)=4\times4^{k+1}+5^{2k+1}=4\times4^{k+1}+4\times5^{2k-1}+25\times5^{2k-1}-4\times5^{2k-1}$ $\text{f}\left(k+1\right)=4\text{f}\left(k\right)+21\times5^{2k-1}$	A1 A1	1.1b 1.1b
	<u>If true for n = k</u> then <u>true for n = k + 1</u>, <u>true for n = 1</u> so <u>true for all (positive integers) n</u> (Allow "for all values")	A1	2.4
		(6)	
(ii) **Way 3**	f(k + 1) – mf(k)		
	When n = 1, $4^{n+1}+5^{2n-1}=16+5=21$ so the statement is true for n = 1	B1	2.2a
	Assume true for n = k so $4^{k+1}+5^{2k-1}$ is divisible by 21	M1	2.4
	$\text{f}\left(k+1\right)-m\text{f}\left(k\right)=4^{k+2}+5^{2k+1}-m\left(4^{k+1}+5^{2k-1}\right)$	M1	2.1
	$=\left(4-m\right)4^{k+1}+5^{2k+1}-m\times5^{2k-1}$ $=\left(4-m\right)\left(4^{k+1}+5^{2k-1}\right)+21\times5^{2k-1}$	A1	1.1b
	$=\left(4-m\right)\left(4^{k+1}+5^{2k-1}\right)+21\times5^{2k-1}+m\text{f}\left(k\right)$	A1	1.1b
	<u>If true for n = k</u> then <u>true for n = k + 1</u>, <u>true for n = 1</u> so <u>true for all (positive integers) n</u> (Allow "for all values")	A1	2.4
		(6)	
(ii) **Way 4**	f(k) = 21M		
	When n = 1, $4^{n+1}+5^{2n-1}=16+5=21$ so the statement is true for n = 1	B1	2.2a
	Assume true for n = k so $4^{k+1}+5^{2k-1}=21M$	M1	2.4
	$\text{f}\left(k+1\right)=4^{k+1+1}+5^{2(k+1)-1}$	M1	2.1
	$\text{f}\left(k+1\right)=4\times4^{k+1}+5^{2k+1}=4\left(21M-5^{2k-1}\right)+5^{2k+1}$ $\text{f}\left(k+1\right)=84M+21\times5^{2k-1}$	A1 A1	1.1b 1.1b
	<u>If true for n = k</u> then <u>true for n = k + 1</u>, <u>true for n = 1</u> so <u>true for all (positive integers) n</u> (Allow "for all values")	A1	2.4
		(6)	
			(12 marks)

Notes

(i)
B1: Shows that the result holds for n = 1. Must see **substitution** into the rhs.
The minimum would be: $\begin{pmatrix} 4+1 & -8 \\ 2 & 1-4 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$.
M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.)
M1: Sets up a correct multiplication statement either way round
A1: Achieves a correct un-simplified matrix
A1: Reaches a correct simplified matrix with no errors **<u>and the correct un-simplified matrix seen previously.</u>** Note that the simplified result may be proved by equivalence.
A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution.
(ii) **Way 1**
B1: Shows that f(1) = 21
M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.)
M1: Attempts f(k + 1) – f(k) or equivalent work
A1: Achieves a correct expression for f(k + 1) – f(k) in terms of f(k)
A1: Reaches a correct expression for f(k + 1) in terms of f(k)
A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution.
Way 2
B1: Shows that f(1) = 21
M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.)
M1: Attempts f(k + 1)
A1: Correctly obtains 4f(k) **or** $21\times5^{2k-1}$
A1: Reaches a correct expression for f(k + 1) in terms of f(k)
A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution.
Way 3
B1: Shows that f(1) = 21
M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.)
M1: Attempts f(k + 1) – mf(k)
A1: Achieves a correct expression for f(k + 1) – mf(k) in terms of f(k)
A1: Reaches a correct expression for f(k + 1) in terms of f(k)
A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution.

Way 4

B1: Shows that f(1) = 21

M1: Makes a statement that assumes the result is true for some value of n (Assume (true for) $n = k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n = k$ then … etc.)

M1: Attempts f(k + 1)

A1: Correctly obtains $84M$ **or** $21\times5^{2k-1}$

A1: Reaches a correct expression for f(k + 1) in terms of M and 5^{2k-1}

A1: Correct conclusion. This mark is dependent on all previous marks apart from the B mark. It is gained by conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution.

3. Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$

(6)

Question	Scheme	Marks	AOs
3	$n=1,\ \sum_{r=1}^{1}\frac{1}{(2r-1)(2r+1)}=\frac{1}{1\times 3}=\frac{1}{3}$ and $\frac{n}{2n+1}=\frac{1}{2\times 1+1}=\frac{1}{3}$ (true for n=1)	B1	2.2a
	Assume general statement is true for $n=k$. So assume $\sum_{r=1}^{k}\frac{1}{(2r-1)(2r+1)}=\frac{k}{2k+1}$ is true.	M1	2.4
	$\left(\sum_{r=1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\right)$ "$\frac{k}{2k+1}$" $+\frac{1}{(2k+1)(2k+3)}$	M1	2.1
	$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$	dM1	1.1b
	$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}=\frac{(k+1)}{2(k+1)+1}$ or $\frac{k+1}{2k+3}$	A1	1.1b
	As $\sum_{r=1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\frac{(k+1)}{2(k+1)+1}$ then the general result is true for $n=k+1$ As the general result has been shown to be <u>true for $n=1$</u>, and <u>true for $n=k$ implies true for $n=k+1$</u>, so the result <u>is true for all n</u> $\in\mathbb{N}$	A1cso	2.4
		(6)	
		(6 marks)	

Notes

	B1	Substitutes $n=1$ into both sides of the statement to show they are eqaul. As a minimum expect to see $\frac{1}{1\times 3}$ and $\frac{1}{2+1}$ for the substitutions. (No need to state true for n = 1 for this mark.)
	M1	Assumes (general result) true for $n=k$. (Assume (true for) n = k is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for n = k then ... etc.)
	M1	Attempts to add $(k+1)$th term to their sum of k terms. Must be adding the (k+1)th term but allow slips with the sum.
	dM1	Depends on previous M. Combines their two fractions over a correct common denominator for their fractions, which may be $(2k+1)^2(2k+3)$ (allow a slip in the numerator).
	A1	Correct algebraic work leading to $\frac{(k+1)}{2(k+1)+1}$ or $\frac{k+1}{2k+3}$
	A1	**cso** Depends on all except the **B** mark being scored (but must have an attempt to show the n = 1 case). Demonstrates the expression is the correct for n = k + 1 (both sides must have been seen somewhere) and gives a correct induction statement with **all** three underlined statements (or equivalents) seen at some stage during their solution (so true for n = 1 may be seen at the start). For demonstrating the correct expression, accept giving in the form $\frac{(k+1)}{2(k+1)+1}$, or reaching $\frac{k+1}{2k+3}$ and stating "which is the correct form with n = k + 1" or similar – but some indication is needed. Note: if mixed variables are used in working (r's and k's mixed up) then withhold the final A. Note: If n is used throughout instead of k allow all marks if earned.

8. Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 2^{n+2} + 3^{2n+1}$$

is divisible by 7

(6)

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question	Scheme	Marks	AOs
8	**Way 1: $\mathbf{f}(\boldsymbol{k}+1)-\mathbf{f}(\boldsymbol{k})$** When $n=1$, $2^{n+2}+3^{2n+1}=2^3+3^3=35$ Shows the statement is true for $n=1$, allow 5(7)	B1	2.2a
	Assume true for $\boldsymbol{n}=\boldsymbol{k}$, so $2^{k+2}+3^{2k+1}$ is divisible by 7	M1	2.4
	$\mathrm{f}(k+1)-\mathrm{f}(k)=2^{k+3}+3^{2k+3}-\left(2^{k+2}+3^{2k+1}\right)$	M1	2.1
	$=2\times2^{k+2}+9\times3^{2k+1}-2^{k+2}-3^{2k+1}$ $=2^{k+2}+8\times3^{2k+1}$ $=\mathrm{f}(k)+7\times3^{2k+1}$ or $8\mathrm{f}(k)-7\times2^{k+2}$	A1	1.1b
	$\mathrm{f}(k+1)=2\mathrm{f}(k)+7\times3^{2k+1}$ or $9\mathrm{f}(k)-7\times2^{k+2}$	A1	1.1b
	If **true for $\boldsymbol{n}=\boldsymbol{k}$** then **true for $\boldsymbol{n}=\boldsymbol{k}+1$** and as it is **true for $\boldsymbol{n}=1$** the statement is **true for all** (positive integers) ***n***	A1	2.4
		(6)	
	Way 2: $\mathbf{f}(\boldsymbol{k}+1)$ When $n=1$, $2^{n+2}+3^{2n+1}=2^3+3^3=35$ So the statement is true for $n=1$	B1	2.2a
	Assume true for $n=k$, so $2^{k+2}+3^{2k+1}$ is divisible by 7	M1	2.4
	$\mathrm{f}(k+1)=2^{(k+1)+2}+3^{2(k+1)+1}$	M1	2.1
	$\mathrm{f}(k+1)=2^{k+3}+3^{2k+3}=2\times2^{k+2}+9\times3^{2k+1}$ $=2\left(2^{k+2}+3^{2k+1}\right)+7\times3^{2k+1}$ $=2\mathrm{f}(k)+7\times3^{2k+1}$or $9\mathrm{f}(k)-7\times2^{k+2}$	A1 A1	1.1b 1.1b
	If true for $\boldsymbol{n}=\boldsymbol{k}$ then **true for $\boldsymbol{n}=\boldsymbol{k}+1$** and as it is **true for $\boldsymbol{n}=1$** the statement is **true for all** (positive integers) ***n***	A1	2.4
		(6)	
	Way 3: $\mathbf{f}(\boldsymbol{k}+1)-\boldsymbol{m}\,\mathbf{f}(\boldsymbol{k})$ When $n=1$, $2^{n+2}+3^{2n+1}=2^3+3^3=35$ So the statement is true for $n=1$	B1	2.2a
	Assume true for $n=k$, so $2^{k+2}+3^{2k+1}$ is divisible by 7	M1	2.4
	$\mathrm{f}(k+1)-m\mathrm{f}(k)=2^{k+3}+3^{2k+3}-m\left(2^{k+2}+3^{2k+1}\right)$	M1	2.1
	$=2\times2^{k+2}+9\times3^{2k+1}-m\times2^{k+2}-m\times3^{2k+1}$ $=(2-m)2^{k+2}+9\times3^{2k+1}-m\times3^{2k+1}$ $=(2-m)\left(2^{k+2}+3^{2k+1}\right)+7\times3^{2k+1}$	A1	1.1b
	$\mathrm{f}(k+1)=(2-m)\left(2^{k+2}+3^{2k+1}\right)+7\times3^{2k+1}+m\mathrm{f}(k)$	A1	1.1b

	If true for $\boldsymbol{n = k}$ then **true for** $\boldsymbol{n = k + 1}$ and as it is **true for** $\boldsymbol{n = 1}$ the statement is **true for all** (positive integers) $\boldsymbol{n}$	A1	2.4
		(6)	
			(6 marks)

Notes:

Way 1: $\mathbf{f}(\boldsymbol{k}+1)-\mathbf{f}(\boldsymbol{k})$

B1: Shows that f(1) = 35 and concludes or shows divisible by 7. This may be seen in the final statement.

M1: Makes a statement that assumes the result is true for some value of *n*

M1: Attempts $\mathrm{f}(k+1)-\mathrm{f}(k)$

A1: Achieves a correct expression for $\mathrm{f}(k+1)-\mathrm{f}(k)$ in terms of $\mathrm{f}(k)$

A1: Reaches a correct expression for $\mathrm{f}(k+1)$ in terms of $\mathrm{f}(k)$

A1: Correct conclusion. This mark is dependent on all previous marks, look out for scoring B1 in this statement. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution.

Way 2: $\mathbf{f}(\boldsymbol{k}+1)$

B1: Shows that f(1) = 35 and concludes divisible by 7

M1: Makes a statement that assumes the result is true for some value of *n*

M1: Attempts $\mathrm{f}(k+1)$

A1: Correctly obtains either $2\mathrm{f}(k)$ **or** $7\times 3^{2k+1}$ or either $9\mathrm{f}(k)$ or $-7\times 2^{k+2}$

A1: Reaches a correct expression for $\mathrm{f}(k+1)$ in terms of $\mathrm{f}(k)$

A1: Correct conclusion. This mark is dependent on all previous marks. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution.

Way 3: $\mathbf{f}(\boldsymbol{k}+1)-\boldsymbol{m}\,\mathbf{f}(\boldsymbol{k})$

B1: Shows that f(1) = 35 and concludes divisible by 7

M1: Makes a statement that assumes the result is true for some value of *n*

M1: Attempts $\mathrm{f}(k+1)-m\mathrm{f}(k)$

A1: Achieves a correct expression for $\mathrm{f}(k+1)-m\mathrm{f}(k)$ in terms of $\mathrm{f}(k)$

A1: Reaches a correct expression for $\mathrm{f}(k+1)$ in terms of $\mathrm{f}(k)$

A1: Correct conclusion. This mark is dependent on all previous marks. It is gained by conveying the ideas of **all four bold** points either at the end of their solution or as a narrative in their solution.

8. (a) Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)^2(n+2)$$

(6)

(b) Hence, show that, for all positive integers n,

$$\sum_{r=n}^{2n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)(an+b)(cn+d)$$

where a, b, c and d are integers to be determined.

(3)

Question	Scheme	Marks	AOs
8(a)	$n=1,\quad \text{lhs}=1(2)(3)=6,\quad \text{rhs}=\frac{1}{2}(1)(2)^2(3)=6$ (true for n = 1)	B1	2.2a
	Assume true for $n = k$ so $\sum_{r=1}^{k} r(r+1)(2r+1)=\frac{1}{2}k(k+1)^2(k+2)$	M1	2.4
	$\sum_{r=1}^{k+1} r(r+1)(2r+1)=\frac{1}{2}k(k+1)^2(k+2)+(k+1)(k+2)(2k+3)$	M1	2.1
	$=\frac{1}{2}(k+1)(k+2)\left[k(k+1)+2(2k+3)\right]$	dM1	1.1b
	$=\frac{1}{2}(k+1)(k+2)\left[k^2+5k+6\right]=\frac{1}{2}(k+1)(k+2)(k+2)(k+3)$ Shows that $=\frac{1}{2}(\underline{k+1})(\underline{k+1}+1)^2(\underline{k+1}+2)$ Alternatively shows that $\sum_{r=1}^{k+1} r(r+1)(2r+1)=\frac{1}{2}(k+1)(k+1+1)^2(k+1+2)$ $=\frac{1}{2}(k+1)(k+2)^2(k+3)$ Compares with their summation and concludes true for $n = k+1$	A1	1.1b
	If the statement is **true for *n* = *k* then** it has been shown **true for *n* = *k* + 1** and as it is **true for *n* = 1**, the statement is **true** for all **positive integers *n*.**	A1	2.4
		(6)	
(b)	$\sum_{r=n}^{2n} r(r+1)(2r+1)=n(2n+1)^2(2n+2)-\frac{1}{2}(n-1)n^2(n+1)$	M1	3.1a
	$=\frac{1}{2}n(n+1)\left[4(2n+1)^2-n(n-1)\right]$	M1	1.1b
	$=\frac{1}{2}n(n+1)(15n^2+17n+4)$ $=\frac{1}{2}n(n+1)(3n+1)(5n+4)$	A1	1.1b
		(3)	
			(9 marks)

Notes

(a) Note ePen B1 M1 M1 A1 A1 A1

B1: Substitutes n = 1 into both sides to show that they are both equal to 6. (There is no need to state true for n = 1 for this mark)

M1: Makes a statement that assumes the result is true for some value of n, say k

M1: Adds the (k + 1)th term to the assumed result

dM1: Dependent on previous M, factorises out $\frac{1}{2}(k+1)(k+2)$

A1: Reaches a correct the required expression no errors and shows that this is the correct sum for $n = k+1$

A1: Depends on all except **B** mark being scored (must have been some attempt to show true for n = 1). Correct conclusion conveying all the points in bold.

(b)

M1: Realises that $\sum_{r=1}^{2n} r(r+1)(2r+1) - \sum_{r=1}^{n-1} r(r+1)(2r+1)$ is required and uses the result from part (a) to obtain the required sum in terms of n

M1: Attempts to factorise by $\frac{1}{2}n(n+1)$

A1: Correct expression or correct values

7. Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\begin{pmatrix} -5 & 9 \\ -4 & 7 \end{pmatrix}^n = \begin{pmatrix} 1-6n & 9n \\ -4n & 1+6n \end{pmatrix}$$

(6)

Question	Scheme	Marks	AOs
7	For $n=1$: $\begin{pmatrix}1-6\times1 & 9\times1\\ -4\times1 & 1+6\times1\end{pmatrix}=\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}=\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}^1$ So the statement is true for $n=1$	B1	2.2a
	Assume true for $\boldsymbol{n=k}$, or **Assume** $\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}^k=\begin{pmatrix}1-6k & 9k\\ -4k & 1+6k\end{pmatrix}$	M1	2.5
	$\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}^{k+1}=\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}^k\times\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}$ OR $\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}\times\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}^k$	M1	2.1
	$=\begin{pmatrix}1-6k & 9k\\ -4k & 1+6k\end{pmatrix}\times\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}=\begin{pmatrix}-5+30k-36k & 9-54k+63k\\ 20k-4-24k & -36k+7+42k\end{pmatrix}$ OR $=\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}\times\begin{pmatrix}1-6k & 9k\\ -4k & 1+6k\end{pmatrix}=\begin{pmatrix}-5+30k-36k & -45k+9+54k\\ -4+24k-28k & -36k+7+42k\end{pmatrix}$	M1	1.1b
	Achieves from fully correct working $=\begin{pmatrix}-5-6k & 9+9k\\ -4-4k & 7+6k\end{pmatrix}$	A1	1.1b
	$=\begin{pmatrix}1-6(k+1) & 9(k+1)\\ -4(k+1) & 1+6(k+1)\end{pmatrix}$ Hence the result is true for $n=k+1$. Since it is <u>true for $n=1$</u>, and <u>if true for $n=k$ then true for $n=k+1$</u>, thus by mathematical induction the <u>result holds for all n</u> $\in\mathbb{N}$	A1cso	2.4
		(6)	
		(6 marks)	

Notes:

(a)

B1: Shows the statement is true for $n=1$. Accept as minimum $\begin{pmatrix}1-6 & 9\\ -4 & 1+6\end{pmatrix}=\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}$

M1: Makes the inductive assumption, **assume** true $\boldsymbol{n=k}$**.** This may appear in the conclusion.

M1: A correct statement for $\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}^{k+1}$ in terms of $\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}^k$, can be either way round.

Can be implied by $\begin{pmatrix}1-6k & 9k\\ -4k & 1+6k\end{pmatrix}\times\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}$ or $\begin{pmatrix}-5 & 9\\ -4 & 7\end{pmatrix}\times\begin{pmatrix}1-6k & 9k\\ -4k & 1+6k\end{pmatrix}$

M1: Carries out the multiplication correctly, condone sign slips

A1: Correct simplified matrix **from fully correct working**

A1: Completes the inductive argument by showing clearly the matrix has the correct form (must have $(k+1)$ factors in terms) or uses the result with $n=k+1$ and shows that their result is the same.

Conclusion conveying **all** three underlined points or equivalent at some point in their argument. Depends on all three M's and A marks but can be scored without the B mark as long as it is stated true for $n=1$