Cp1Ch6 XMQs and MS

(Total: 109 marks)

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 1. CP1_Specimen Q2 .  8 marks - CP1ch6 Matrices
 2. CP2_2019     Q7 . 11 marks - CP1ch6 Matrices
 3. CP2_2020     Q6 . 14 marks - CP1ch6 Matrices
 4. CP1_2021     Q4 .  9 marks - CP1ch6 Matrices
 5. CP1_2022     Q5 .  6 marks - CP1ch6 Matrices
 6. CP2_2022     Q2 .  8 marks - CP1ch6 Matrices
 7. CP(AS)_2018  Q1 .  5 marks - CP1ch6 Matrices
 8. CP(AS)_2019  Q10. 12 marks - CP1ch6 Matrices
 9. CP(AS)_2020  Q1 .  6 marks - CP1ch6 Matrices
10. CP(AS)_2020  Q6 . 16 marks - CP1ch6 Matrices
11. CP(AS)_2021  Q4 .  7 marks - CP1ch6 Matrices
12. CP(AS)_2022  Q1 .  7 marks - CP1ch6 Matrices
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2. A company runs three theme parks, A (Aztec Adventureland), B (Babylonian Towers) and C (Carthaginian Kingdom).

It is known that park A makes a profit of £30 per visitor, park B makes a profit of £26 per visitor and park C makes a profit of £33 per visitor.

In 2017 the Aztec Adventureland park was upgraded, which took one year to carry out. During 2017

- park A had only 50% of the number of visitors it had in 2016
- park B had 25% more than the number of visitors it had in 2016
- park C had 15% more than the number of visitors it had in 2016

In total 1 350 000 people visited the three theme parks during 2017.

The company made a total profit from the parks of £39.15 million in 2016. The profits dropped by 1% for 2017.

Form and solve a matrix equation to find, to 2 significant figures, the number of visitors for each of the theme parks in 2016.

(8)

Question	Scheme	Marks	AOs
2	Profit in 2017 is 0.99×£39.15m = £38.7585m	B1	2.2a
	Let x = number of visitors to park A in 2016, y = number of visitors to park B in 2016 and z = number of visitors to park C in 2016. So $0.5x + 1.25y + 1.15z = 1.35\times10^6$ $30x + 26y + 33z = 39.15\times10^6$ $15x + 32.5y + 37.95z = 38.7585\times10^6$	M1 A1	3.1b 1.1b
	Hence $\begin{pmatrix}0.5 & 1.25 & 1.15\\ 30 & 26 & 33\\ 15 & 32.5 & 37.95\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1.35\times10^6\\ 39.15\times10^6\\ 38.7585\times10^6\end{pmatrix}$	M1 A1	3.1a 1.1b
	$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0.5 & 1.25 & 1.15\\ 30 & 26 & 33\\ 15 & 32.5 & 37.95\end{pmatrix}^{-1}\begin{pmatrix}1.35\times10^6\\ 39.15\times10^6\\ 38.7585\times10^6\end{pmatrix}=\begin{pmatrix}\ldots\\ \ldots\\ \ldots\end{pmatrix}$	M1	1.1b
	$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}400331.75...\\ 593257.41...\\ 355010.74...\end{pmatrix}$	A1	1.1b
	So in 2016 park A had 400 000 visitors, park B had 590 000 visitors and park C had 360 000 visitors to 2 s.f.	A1ft	3.2a
		(8)	
			(8 marks)

Notes:

B1: Deduces correct total profit for 2017.

M1: Attempts to set up equations in the three unknowns using the information given.

A1: At least two equations correct, with appropriate variables defined.

M1: Sets up a matrix equation from their three equations.

A1: Correct matrix equation (or equivalent).

M1: Solves the equation using the inverse of their matrix (found from calculator or otherwise) to obtain at least one value of x, y or z.

A1: Correct answer as a vector.

A1ft: Interprets their answer in the context of the question with figures quoted to 2 s.f. Withhold this mark if answers not given to 2 s.f.

Note Accept equivalents throughout with the rows in a different order.

Note If the $\times10^6$ is missing throughout, allow the Ms but not the As as they have not interpreted the context (but if units are later sorted out and correct answer reached, full marks can be awarded).

Note $\begin{pmatrix}0.5 & 1.25 & 1.15\\ 30 & 26 & 33\\ 15 & 32.5 & 37.95\end{pmatrix}^{-1}=\begin{pmatrix}0.4916... & 0.0576... & -0.0650...\\ 3.6871... & -0.0098... & -0.1031...\\ -3.3519... & -0.0143... & 0.1403...\end{pmatrix}$

7.

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{pmatrix} \qquad \text{where } k \text{ is a constant}$$

(a) Find the values of k for which the matrix **M** has an inverse.

(2)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect

$$2x - y + z = p$$
$$3x - 6y + 4z = 1$$
$$3x + 2y - z = 0$$

(5)

(c) (i) Find the value of q for which the set of simultaneous equations

$$2x - y + z = 1$$
$$3x - 5y + 4z = q$$
$$3x + 2y - z = 0$$

can be solved.

(ii) For this value of q, interpret the solution of the set of simultaneous equations geometrically.

(4)

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Question	Scheme	Marks	AOs
7(a)	$\lvert\mathbf{M}\rvert = 2(-k-8)+1(-3-12)+1(6-3k)=0 \Rightarrow k=\ldots$	M1	1.1b
	$k \neq -5$	A1	2.4
		(2)	
(b) **Way 1**	$\mathbf{M}=\begin{pmatrix}2 & -1 & 1\\ 3 & -6 & 4\\ 3 & 2 & -1\end{pmatrix} \Rightarrow \begin{pmatrix}x\\ y\\ z\end{pmatrix}=\mathbf{M}^{-1}\begin{pmatrix}p\\ 1\\ 0\end{pmatrix}$	M1	3.1a
	$\mathbf{M}^{-1}=\frac{1}{5}\begin{pmatrix}-2 & 1 & 2\\ 15 & -5 & -5\\ 24 & -7 & -9\end{pmatrix}$	B1	1.1b
	$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-2 & 1 & 2\\ 15 & -5 & -5\\ 24 & -7 & -9\end{pmatrix}\begin{pmatrix}p\\ 1\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}x\\ y\\ z\end{pmatrix}=\ldots$	M1	2.1
	$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-2p+1\\ 15p-5\\ 24p-7\end{pmatrix}$	A1	1.1b
	$\left(\frac{-2p+1}{5},\ 3p-1, \frac{24p-7}{5}\right)$	A1ft	2.5
		(5)	
(b) **Way 2**	$\begin{matrix}2x-\ y+\ z=p\\ 3x-6y+4z=1\\ 3x+2y-\ z=0\end{matrix} \Rightarrow \text{e.g.} \begin{matrix}8y-5z=-1\\ 9y-5z=3p-2\end{matrix} \Rightarrow y=\ldots$ $\Rightarrow x=\ldots, z=\ldots$	M1	3.1a
	$y=3p-1$(or $x=\frac{-2p+1}{5}$ or $z=\frac{24p-7}{5}$)	B1	1.1b
	$8(3p-1)-5z=-1 \Rightarrow z=\ldots \Rightarrow x=\ldots$	M1	2.1
	$z=\frac{24p-7}{5}$, $x=\frac{-2p+1}{5}$	A1	1.1b
	$\left(\frac{-2p+1}{5},\ 3p-1, \frac{24p-7}{5}\right)$	A1ft	2.5

(c)(i)	For consistency: E.g. $5x+y=4-q$ and $15x+3y=q$	M1	3.1a
	$4-q=\frac{q}{3}\Rightarrow q=...$	M1	2.1
	$q=3$	A1	1.1b
	Alternative for (c)(i): $x=1\Rightarrow 2-y+z=1,\ 3+2y-z=0\Rightarrow y=...,z=...$ M1 for allocating a number to one variable and solves for the other 2 $x=1, y=-4, z=-5\Rightarrow 3+20-20=q$ M1 substitutes into the second equation and solves for *q* A1: *q* = 3		
(ii)	Three **planes** that intersect in a **line** Or Three **planes** that form a **sheaf** allow **sheath**!	B1	2.4
		(4)	
			(11 marks)

Notes

(a)

M1: Attempts determinant, equates to zero and attempts to solve for *k* in order to establish the restriction for *k*. For the determinant, at least 2 of the 3 "elements" should be correct.

May see rule of Sarrus used for determinant e.g.

$$|\mathbf{M}|=(2)(k)(-1)+(4)(3)(-1)+(3)(2)(1)-(3)(k)(-1)-(2)(4)(2)-(-1)(3)(-1)=0\Rightarrow k=...$$

A1: Describes the correct condition for *k* with no contradictions. Allow e.g. $k < -5$, $k > -5$

(b)**Way 1**

M1: A complete strategy for solving the given equations. Need to see an attempt at the inverse followed by a correct method for finding *x*, *y* and *z*

B1: Correct inverse matrix

M1: Uses their inverse and attempts the multiplication with the correct vector

A1: Correct values for *x*, *y* and *z* in any form

A1ft: Correct values given in coordinate form only. **Follow through their *x*, *y* and *z*.**

Way 2

M1: A complete strategy for solving the given equations. Need to see an attempt at eliminating one variable followed by a correct method for finding *x*, *y* and *z*

B1: One correct value

M1: Uses the equations to find values for the other 2 variables

A1: Correct values for *x*, *y* and *z* in any form

A1ft: Correct values given in coordinate form only. **Follow through their *x*, *y* and *z*.**

(c)(i)

M1: Uses a correct strategy that will lead to establishing a value for *q*. E.g. eliminating one of *x*, *y* or *z*

M1: Solves a suitable equation to obtain a value for *q*

A1: Correct value

(ii)

B1: Describes the correct geometrical configuration.

Must include the **two** ideas of **planes** and meeting in a **line** or forming a **sheaf** with no contradictory statements.

6.

$$\mathbf{M} = \begin{pmatrix} k & 5 & 7 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \qquad \text{where } k \text{ is a constant}$$

(a) Given that $k \neq 4$, find, in terms of k, the inverse of the matrix $\mathbf{M}$.

(4)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect.

$$\begin{aligned} 2x + 5y + 7z &= 1 \\ x + y + z &= p \\ 2x + y - z &= 2 \end{aligned}$$

(3)

(c) (i) Find the value of q for which the following planes intersect in a straight line.

$$\begin{aligned} 4x + 5y + 7z &= 1 \\ x + y + z &= q \\ 2x + y - z &= 2 \end{aligned}$$

(ii) For this value of q, determine a vector equation for the line of intersection.

(7)

Question	Scheme	Marks	AOs
6(a)	$\lvert\mathbf{M}\rvert = k(-1-1)-5(-1-2)+7(1-2)$ $\{=8-2k\}$	M1	1.1b
	Minors: $\begin{pmatrix} -2 & -3 & -1 \\ -12 & -k-14 & k-10 \\ -2 & k-7 & k-5 \end{pmatrix}$ Cofactors: $\begin{pmatrix} -2 & 3 & -1 \\ 12 & -k-14 & 10-k \\ -2 & 7-k & k-5 \end{pmatrix}$	M1	1.1b
	$\mathbf{M}^{-1} = \dfrac{1}{8-2k}\begin{pmatrix} -2 & 12 & -2 \\ 3 & -k-14 & 7-k \\ -1 & 10-k & k-5 \end{pmatrix}$	M1 A1	2.1 1.1b
		(4)	
(b)	$\mathbf{M}^{-1} = \dfrac{1}{4}\begin{pmatrix} -2 & 12 & -2 \\ 3 & -16 & 5 \\ -1 & 8 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} 1 \\ p \\ 2 \end{pmatrix}$ Solve the equations simultaneously to achieve values for *x*, *y* and *z* $y+3z=2p-2$ and $4y+8z=-1$ Þ $x=\ldots,\ y=\ldots,\ z=\ldots$	M1	3.1a
	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}+3p-1 \\ \frac{3}{4}-4p+\frac{5}{2} \\ -\frac{1}{4}+2p-\frac{3}{2} \end{pmatrix}$	A1ft	1.1b
	$\left(\dfrac{12p-6}{4}, \dfrac{13-16p}{4}, \dfrac{8p-7}{4}\right)$ $\left(3p-\dfrac{3}{2}, \dfrac{13}{4}-4p, 2p-\dfrac{7}{4}\right)$	A1	2.2a
		(3)	
(c)(i)			
	For consistency: E.g. eliminates *z* to find two equations from $3x+2y=q+2$ $3x+2y=7q-1$ $18x+12y=15$ For consistency: E.g. eliminates *x* to find two equations from $y+3z=1-4q$ $3y+9z=-3$ $y+3z=2q-2$ For consistency: E.g. eliminates *y* to find two equations from $x-2z=2-q$ $-x+2z=1-5q$ $-6x+12z=-9$	M1	3.1a
	e.g. $q+2=7q-1 \Rightarrow q=\ldots$ e.g. $-3=3(1-4q) \Rightarrow q=\ldots$ e.g. $-9=6(1-5q) \Rightarrow q=\ldots$	M1	1.1b
	$q=\dfrac{1}{2}$	A1	1.1b

	Alternative Equating coefficients leading to two out of three equations and solves to find values for a and b $4a+b=2, 5a+b=1, 7a+b=-1$ $\{a=-1, b=6\}$	M1	3.1a
	Forms the fourth equation involving q $a+bq=2$ and substitutes in the values of a and b to finds a value for q	M1	1.1b
	$q=\frac{1}{2}$	A1	1.1b
	Finds a coordinate of intersection of the planes $4x+5y+7z=1$ and $2x+y-z=2$ e.g let $z=0 \Rightarrow 4x+5y=1$ and $2x+y=2 \Rightarrow y=-1, x=1.5$	M1	3.1a
	Substitutes the values for x, y and z into $x+y+z=q$ to reach a value for q	M1	1.1b
	$q=\frac{1}{2}$	A1	1.1b
(ii)	For example: $x=\lambda \Rightarrow 3\lambda+2y=\frac{5}{2},\ \lambda-2z=\frac{3}{2} \Rightarrow y=\mathrm{f}(\lambda), z=\mathrm{f}(\lambda)$ $y=\lambda \Rightarrow 3x+2\lambda=\frac{5}{2},\ \lambda+3z=1 \Rightarrow x=\mathrm{f}(\lambda), z=\mathrm{f}(\lambda)$ $z=\lambda \Rightarrow 3y+9\lambda=-3,\ -6x+12\lambda=-9 \Rightarrow x=\mathrm{f}(\lambda), y=\mathrm{f}(\lambda)$	M1	3.1a
	Let $x=\lambda$, $\lambda=\frac{y-\frac{5}{4}}{-\frac{3}{2}}=\frac{z+\frac{3}{4}}{\frac{1}{2}}$ or $y=\frac{5}{4}-\frac{3}{2}\lambda, z=-\frac{3}{4}+\frac{1}{2}\lambda$ Let $y=\lambda$, $\lambda=\frac{x-\frac{5}{6}}{-\frac{2}{3}}=\frac{z+\frac{1}{3}}{-\frac{1}{3}}$ or $x=\frac{5}{5}-\frac{2}{3}\lambda, z=-\frac{1}{3}-\frac{1}{3}\lambda$ Let $z=\lambda$, $\lambda=\frac{x-\frac{3}{2}}{2}=\frac{y+1}{-3}$ or $x=\frac{3}{2}+2\lambda, y=-1-3\lambda$	A1	1.1b
	$\mathbf{r}=\frac{5}{4}\mathbf{j}-\frac{3}{4}\mathbf{k}+t(2\mathbf{i}-3\mathbf{j}+\mathbf{k})$ o.e. $\mathbf{r}=\frac{5}{6}\mathbf{i}-\frac{1}{3}\mathbf{k}+t(2\mathbf{i}-3\mathbf{j}+\mathbf{k})$ o.e. $\mathbf{r}=\frac{3}{2}\mathbf{i}-\mathbf{j}+t(2\mathbf{i}-3\mathbf{j}+\mathbf{k})$ o.e.	M1 A1	1.1b 2.5
	Alternative (ii) Finds two different coordinates that lie on the line of intersection For example: setting $x=0 \Rightarrow \left(0,\frac{5}{4},-\frac{3}{4}\right)$ setting $y=0 \Rightarrow \left(\frac{5}{6},0,-\frac{1}{3}\right)$ setting $z=0 \Rightarrow \left(\frac{3}{2},-1,0\right)$	M1 A1	3.1a 1.1b
	Finds the vector equation of the line passing through their two points	M1	1.1b
	$\mathbf{r}=\frac{5}{4}\mathbf{j}-\frac{3}{4}\mathbf{k}+t(2\mathbf{i}-3\mathbf{j}+\mathbf{k})$ o.e.	A1	2.5

	$\mathbf{r} = \frac{5}{6}\mathbf{i} - \frac{1}{3}\mathbf{k} + t(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$ o.e. $\mathbf{r} = \frac{3}{2}\mathbf{i} - \mathbf{j} + t(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$ o.e.		
	Alternative (ii): Outside the spec Finds the cross product of two normal vectors and a coordinate that lies on all three planes	M1	3.1a
	Correct cross product $\begin{vmatrix} 4 & 5 & 7 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ or $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \begin{pmatrix} 12 \\ 18 \\ -3 \end{pmatrix}$	A1	1.1b
	Uses the point and the direction vector to find the equation of the line	M1	1.1b
	$\mathbf{r} = \frac{5}{4}\mathbf{j} - \frac{3}{4}\mathbf{k} + t(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$ o.e.	A1	1.1b
		(7)	
			(14 marks)

Notes

(a)
M1: Correct method to find the determinant. Condone one sign slip
M1: A correct first step in obtaining the inverse. Could be the matrix of minors or cofactors. Condone sign slips as long as the intention is clear.
M1: Fully correct method to obtain the inverse. Attempts matrix of minors, cofactors, transposes and 1/determinant
A1: Correct matrix
(b)
M1: A complete strategy for solving the given equations e.g. multiplies the given coordinates by their inverse or solves simultaneously to achieve values for x, y and z
A1ft: Correct calculation on their inverse matrix (unsimplified) or at least on correct value if solving simultaneously
A1: Correct coordinates
(c)(i)
M1: Uses a correct strategy that will lead to establishing a value for q. E.g. eliminating one of x, y or z
M1: Solves a suitable equation to obtain a value for q
A1: Correct value
(c)(i) **Alternative 1**
M1: Equating coefficients leading to two out of three equations and solves to find values for a and b
M1: Solves a suitable equation to obtain a value for q using their values for a and b
(c)(i) **Alternative 2**
M1: Finds a coordinate of intersection of the planes $4x + 5y + 7z = 1$ and $2x + y - z = 2$
M1: Substitutes the values for x, y and z into $x + y + z = q$ to reach a value for q
A1: Correct value
(ii)
M1: Uses a correct strategy to obtain the Cartesian equation of the line or the general coordinates
A1: Correct Cartesian equation or coordinates in terms of a parameter.

M1: Uses their Cartesian equation to correctly extract the position and direction to form a vector equation for the required line
A1: Correct equation (o.e.) look out for multiples of the direction vector
Alternative (ii)
M1: Finds two different coordinates that lie on the line of intersection
A1: Correct coordinates
M1: Uses their coordinates to find the vector equation of the line that passes through them.
A1: Correct equation (o.e.), look out for multiples of the direction vector. Must have **r** = …
Alternative (ii) outside spec
M1: Finds the cross product between the normal vectors of two of the planes and a coordinate that lies on all three planes. If a coordinate is found in (i) it must be used in this part to award this mark.
A1: Correct cross product
M1: Uses the coordinate and the cross product to find the equation of the line
A1: Correct equation (o.e.), look out for multiples of the direction vector. Must have **r** = …

4. (i) **A** is a 2 by 2 matrix and **B** is a 2 by 3 matrix.

Giving a reason for your answer, explain whether it is possible to evaluate

(a) **AB**

(b) **A** + **B**

(2)

(ii) Given that

$$\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix} = \lambda \mathbf{I}$$

where a, b and λ are constants,

(a) determine

- the value of λ
- the value of a
- the value of b

(b) Hence deduce the inverse of the matrix $\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}$

(3)

(iii) Given that

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & \cos\theta \\ 0 & \cos 2\theta & \sin 2\theta \end{pmatrix} \qquad \text{where } 0 \leqslant \theta < \pi$$

determine the values of θ for which the matrix **M** is singular.

(4)

Question	Scheme	Marks	AOs
4(i) (a)	It is possible as the number of columns of matrix **A** matches the number of rows of matrix **B**.	B1	2.4
(b)	It is not possible as matrix **A** and matrix **B** have different dimensions o.e. different number of columns	B1	2.4
		(2)	
(ii) (a)	$\lambda = 5$	B1	2.2a
	$a = 1,\ b = 2$	B1	2.2a
(b)	Inverse matrix $= \frac{1}{5}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix}$	B1 ft	3.1a
		(3)	
(iii)	A complete method to find the determinant of the matrix and set equal to zero.	M1	1.1b
	Determinant = $1(\sin\theta\sin 2\theta - \cos\theta\cos 2\theta) - 1(0) + 1(0) = 0$	A1	1.1b
	Uses compound angle formula to achieve $\cos 3\theta = 0$ leading to $\theta = ...$ or use of $\sin 2q = 2\sin q\cos q$ and $\cos 2q = 1 - 2\sin^2 q$ (e.g. to achieve $\cos q(4\sin^2 q - 1) = 0$) leading to $\theta = ...$ or use of $\sin 2q = 2\sin q\cos q$ and $\cos 2q = 2\cos^2 q - 1$ (e.g. to achieve $4\cos^3 q - 3\cos q = 0$) leading to $\theta = ...$	M1	3.1a
	$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$	A1	1.1b
		(4)	
			(9 marks)

Notes:

(i)(a)

B1: Comments that the number of columns of matrix **A** (2) equals the number of rows of matrix **B** (2) therefore it is possible. Accept other terminology that is clear in intent e.g. “length of **A**” and “height of **B**”

(b)

B1: Comments that matrix **A** and matrix **B** have different dimensions therefore it is not possible.

(ii)(a)

B1: Deduces the correct value for $\lambda = 5$

B1: Deduces the correct values for *a* and *b*

(b)

B1ft: Identifies and applies a correct method find the inverse matrix. May multiply from the given equation, in which case follow through on their value of lambda. Alternatively, award for a correct matrix found by calculator or long hand having found *a* and *b* and using these values in the matrix.

(iii)

M1: A complete method to find the determinant of the matrix and sets it equal to 0

A1: Correct equation

M1: Uses appropriate correct trig identities to solve the equation and finds a value for q

A1: All three correct values $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ and no others in the range.

5.

$$\mathbf{M} = \begin{pmatrix} a & 2 & -3 \\ 2 & 3 & 0 \\ 4 & a & 2 \end{pmatrix} \qquad \text{where } a \text{ is a constant}$$

(a) Show that **M** is non-singular for all values of a.

(2)

(b) Determine, in terms of a, $\mathbf{M}^{-1}$

(4)

P 7 1 7 7 6 A 0 1 2 3 2

Question	Scheme	Marks	AOs
5(a)	$det(\boldsymbol{M}) = a(6) - 2(4) - 3(2a - 12)$	M1	1.1b
	$det(\boldsymbol{M}) = 28 \neq 0$ therefore, non-singular for all values of a	A1	2.4
		(2)	
(b)	Finds the matrix of minors $\begin{pmatrix} 6 & 4 & 2a-12 \\ 4+3a & 2a+12 & a^2-8 \\ 9 & 6 & 3a-4 \end{pmatrix}$	M1	1.1b
	Finds the matrix of cofactors and transposes. $\begin{pmatrix} 6 & -4-3a & 9 \\ -4 & 2a+12 & -6 \\ 2a-12 & 8-a^2 & 3a-4 \end{pmatrix}$	M1	1.1b
	$\frac{1}{28}\begin{pmatrix} 6 & -4-3a & 9 \\ -4 & 2a+12 & -6 \\ 2a-12 & 8-a^2 & 3a-4 \end{pmatrix}$	M1 A1	1.1b 2.1
		(4)	
		(6 marks)	

Notes:

(a)

M1: Finds the determinant of the matrix **M**. Must be seen in part (a). Allow one slip if no method shown.

A1: Correct value for determinant, states doesn't equal 0 (accept > 0) and draws the conclusion that the matrix is non-singular. If non-singular meaning determinant is non-zero is given in a preamble then accept a minimal conclusion (e.g. "hence shown"), but there must be a conclusion.

(b)

M1: Finds the matrix of minors, at least 5 correct values.

M1: Finds the matrix of cofactors and transposes (in either order). Note: some will do all these steps in one go, which is fine as long as it is clear what they have done. Allow minor slips if the process is clearly correct.

M1: Completes the process to find the inverse matrix, divides by the determinant.

A1: Correct matrix.

2. In this question you must show all stages of your working.

A college offers only three courses: Construction, Design and Hospitality.

Each student enrols on just one of these courses.

In 2019, there was a total of 1110 students at this college.

There were 370 more students enrolled on Construction than Hospitality.

In 2020 the number of students enrolled on

- Construction **increased** by 1.25%
- Design **increased** by 2.5%
- Hospitality **decreased** by 2%

In 2020, the total number of students at the college increased by 0.27% to 2 significant figures.

(a) (i) Define, for each course, a variable for the number of students enrolled on that course in 2019.

(ii) Using your variables from part (a)(i), write down **three** equations that model this situation.

(4)

(b) By forming and solving a matrix equation, determine how many students were enrolled on each of the three courses in 2019.

(4)

Question	Scheme	Marks	AOs
2(a)(i)	*x* / *C* = number of **Construction** students *y* / *D* = number of **Design** students *z* / *H* = number of **Hospitality** students	B1	3.3
(ii)	The increase in number of students in 2020 $1110 \times 0.0027\{= 2.997 \approx 3\}$ Or The number of students in 2020 $1110 \times 1.0027 = \{1112.997 \approx 1113\}$	M1	1.1b
	$x + y + z = 1110$ $\quad C + D + H = 1110$ $x - z = 370$ o.e. $\quad C - H = 370$ o.e. $0.0125C + 0.025D - 0.02H = 3$or 2.997 o.e $1.0125C + 1.025D + 0.98H = 1113$ or 1112.997o.e. $0.0125x + 0.025y - 0.02z = 3$ or 2.997o.e$1.0125x + 1.025y + 0.98z = 1113$ or 1112.997 o.e.	M1 A1	3.3 1.1b
		(4)	
(b)	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.0125 & 1.025 & 0.98 \end{pmatrix}\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1110 \\ 370 \\ 1113 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0.0125 & 0.025 & -0.02 \end{pmatrix}\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1110 \\ 370 \\ 3 \end{pmatrix}$	M1 A1ft	1.1b 1.1b
	$\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.0125 & 1.025 & 0.98 \end{pmatrix}^{-1}\begin{pmatrix} 1110 \\ 370 \\ 1113 \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix}$ or $\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0.0125 & 0.025 & -0.02 \end{pmatrix}^{-1}\begin{pmatrix} 1110 \\ 370 \\ 3 \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix}$	dM1	1.1b
	So in 2019, **720** students studied **C**onstruction, **40** students studied **D**esign and **350** students studied **H**ospitality	A1	3.2a
		(4)	
			(8 marks)

Notes:

Mark (i) and (ii) together

(a)(i)

B1: Defines 3 variables, minimum e.g. construction = *C*, Design = *D*, Hospitality = *H*. This may be seen in text of the question, abbreviations may be used

(ii)

M1: Finds either the increase or the number of students in 2020. This may be implied by any equation which equals 1113 or 1112.997. If students use 1100 instead of 1110 this is slip and we can award this mark.

M1: Attempts to use the model to set up at least 2 equations

A1: All 3 simplified equations correct (decimals or fractions), one for each different piece of information. Award with mark even if B0 is scored and it is clear what the variables used stand for. Ignore any additional equations even if incorrect. As soon as 3 correct equations are seen you may award this mark.

Alternative approach

(i) B1: Construction = H + 370, Design = D, Hospitality = H

(ii) M1M1A1: $H + 370 + D + H = 1110$ o.e $C = H + 3701.0125(H + 370) + 1.025D + 0.98H = 1113$ or 1112.997 o.e. they do not need to be simplified

(b) This is M1 M1 A1 A1 on ePen but is marked M1A1M1A1

M1: Uses their equation in part(a) to **s**et up a matrix equation of the form $\begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \ldots & \ldots & \ldots \end{pmatrix}\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \ldots \end{pmatrix}$, where "…" are numerical values.

A1ft: Correct matrix equation for their equations

dM1: Dependent on previous method mark. Writes $\left(\text{their } A()^{-1}\begin{pmatrix} 1110 \\ \text{their "370"} \\ \text{their "3"} \end{pmatrix}\right)$ and obtains at least one value of C, D or H. The inverse matrix need not be found, writing $\mathbf{A}^{-1}\begin{pmatrix} 1110 \\ 370 \\ \text{their "3"} \end{pmatrix} = \ldots$ is sufficient. A correct matrix equation followed by correct values implies this mark.

Condone $\begin{pmatrix} 1110 \\ \text{their "370"} \\ \text{their "3"} \end{pmatrix}\mathbf{A}^{-1} = \ldots$ as long as they reach some values. The values imply the correct method

Note: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0.0125 & 0.025 & -0.02 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{10}{23} = 0.43\ldots & \frac{18}{23} = 0.78\ldots & -\frac{400}{23} = -17.39\ldots \\ \frac{3}{23} = 0.13\ldots & -\frac{13}{23} = -0.56\ldots & \frac{800}{23} = 34.78\ldots \\ \frac{10}{23} = 0.43\ldots & -\frac{5}{23} = -0.21\ldots & -\frac{400}{23} = -17.39\ldots \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.0125 & 1.025 & 0.98 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{410}{23} = 17.82\ldots & \frac{18}{23} = 0.78\ldots & -\frac{400}{23} = -17.39\ldots \\ -\frac{797}{23} = -34.65\ldots & -\frac{13}{23} = -0.56\ldots & \frac{800}{23} = 34.78\ldots \\ \frac{410}{23} = 17.82\ldots & -\frac{5}{23} = -0.21\ldots & -\frac{400}{23} = -17.39\ldots \end{pmatrix}$$

A1: Interprets the answer in the context of the question, minimum is C = 720, D = 40, H = 350 with their variables. Condone the variable not been defined for this mark if it is clear which variable belong to what course.

Note: they must be using a matrix equation to solve the equation to score any marks.

<u>Alternative approach</u>
For example
Equations simplifies to $C - H = 370$, $D + 2H = 740$ and $1.025D + 1.9925H = 738.375$

which leads to $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 1.025 & 1.9925 \end{pmatrix}\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 740 \\ 370 \\ 738.375 \end{pmatrix}$ then $\begin{pmatrix} C \\ D \\ H \end{pmatrix} =$

$\begin{pmatrix} 17.826 & 1 & -17.3913 \\ -34.6521 & 0 & 34.7826 \\ 17.826 & 0 & -17.3913 \end{pmatrix}\begin{pmatrix} 740 \\ 370 \\ 738.375 \end{pmatrix} = \begin{pmatrix} 720 \\ 40 \\ 350 \end{pmatrix}$

Note: A 2 x 2 matrix is fine if it is appropriate for their equation.

<u>Special Case:</u> Forming an equation in one variable
(a)(i) B1: Hospitality = x, Construction = x + 370, Design = 740 – 2x
(ii) M1M1A1: $1.0125(x + 370) + 1.025(740 - 2x) + 0.98x = 1113$ or 1112.997

(a)(i) B1: Hospitality = x – 370, Construction = x , Design = 1480 – 2x
(ii) M1M1A1: $1.0125(x) + 1.025(1480 - 2x) + 0.98(x - 370) = 1113$ or 1112.997

(b) M0A0M0A0: They have an equation and are not forming and solving a matrix equation

Answer ALL questions. Write your answers in the spaces provided.

1.

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & -3 \\ 4 & -2 & 1 \\ 3 & 5 & -2 \end{pmatrix}$$

(a) Find $\mathbf{M}^{-1}$ giving each element in exact form.

(2)

(b) Solve the simultaneous equations

$$2x + y - 3z = -4$$
$$4x - 2y + z = 9$$
$$3x + 5y - 2z = 5$$

(2)

(c) Interpret the answer to part (b) geometrically.

(1)

Question	Scheme	Marks	AOs
1(a)	$\mathbf{M}^{-1}=\frac{1}{69}\begin{pmatrix}1 & 13 & 5\\ -11 & -5 & 14\\ -26 & 7 & 8\end{pmatrix}$	B1 B1	1.1b 1.1b
		(2)	
(b)	$\frac{1}{69}\begin{pmatrix}1 & 13 & 5\\ -11 & -5 & 14\\ -26 & 7 & 8\end{pmatrix}\begin{pmatrix}-4\\ 9\\ 5\end{pmatrix}=\ldots$	M1	1.1b
	$x=2, y=1, z=3$ or $(2,1,3)$ or $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$ or $\begin{pmatrix}2\\ 1\\ 3\end{pmatrix}$	A1	1.1b
		(2)	
(c)	The **point** where three **planes** meet	B1ft	2.2a
		(1)	
			(5 marks)

Notes

(a)

B1: Evidence that the determinant is ± 69 (may be implied by their matrix e.g. where entries are not in exact form: $\pm\begin{pmatrix}0.014 & 0.188 & 0.072\\ -0.159 & -0.072 & 0.203\\ -0.377 & 0.101 & 0.116\end{pmatrix}$)(Should be mostly correct)

Must be seen in part (a).

B1: Fully correct inverse with all elements in **<u>exact</u>** form

(b)

M1: Any complete method to find the values of *x*, *y* and *z* (Must be using **their inverse** if using the method in the main scheme)

A1: Correct coordinates

A solution not using the inverse requires a complete method to find values for *x*, *y* and *z* for the method mark.

Correct coordinates only scores both marks.

(c)

B1: Describes the correct geometrical configuration.

Must include the two ideas of **planes** and **meet in a point** with no contradictory statements.

This is dependent on having obtained a unique point in part (b)

10. The population of chimpanzees in a particular country consists of juveniles and adults. Juvenile chimpanzees do not reproduce.

In a study, the numbers of juvenile and adult chimpanzees were estimated at the start of each year. A model for the population satisfies the matrix system

$$\begin{pmatrix} J_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix} \begin{pmatrix} J_n \\ A_n \end{pmatrix} \qquad n = 0, 1, 2, \ldots$$

where a is a constant, and J_n and A_n are the respective numbers of juvenile and adult chimpanzees n years after the start of the study.

(a) Interpret the meaning of the constant a in the context of the model.

(1)

At the start of the study, the total number of chimpanzees in the country was estimated to be 64 000

According to the model, after one year the number of juvenile chimpanzees is 15 360 and the number of adult chimpanzees is 43 008

(b) (i) Find, in terms of a

$$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1}$$

(3)

(ii) Hence, or otherwise, find the value of a.

(3)

(iii) Calculate the change in the number of juvenile chimpanzees in the first year of the study, according to this model.

(2)

Given that the number of juvenile chimpanzees is known to be in decline in the country,

(c) comment on the short-term suitability of this model.

(1)

A study of the population revealed that adult chimpanzees stop reproducing at the age of 40 years.

(d) Refine the matrix system for the model to reflect this information, giving a reason for your answer.

(*There is no need to estimate any unknown values for the refined model, but any known values should be made clear*.)

(2)

Question	Scheme	Marks	AOs
10. (a)	*a* represents the proportion of juvenile chimpanzees that (survive and) **remain** juvenile chimpanzees the next year.	B1	3.4
		(1)	
(b)(i)	Determinant = $0.82a - 0.08\times0.15$	M1	1.1b
	$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1} = ...\begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}$	M1	1.1b
	$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1} = \frac{1}{0.82a-0.012}\begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}$	A1	1.1b
		(3)	
(ii)	$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1}\begin{pmatrix} 15360 \\ 43008 \end{pmatrix} = \frac{1}{0.82a-0.012}\begin{pmatrix} 0.82\times15360-0.15\times43008 \\ (-0.08)\times15360+43008a \end{pmatrix}$ OR forms equations $15360 = aJ_0 + 0.15\times A_0$ $43008 = 0.08\times J_0 + 0.82\times A_0$	M1	3.1a
	$\frac{1}{0.82a-0.012}\left[6144+\left(43008a-1228.8\right)\right] = 64000$ $\Rightarrow 4915.2+43008a = 64000(0.82a-0.012) \Rightarrow a = ...$ OR $A_0 = 64000 - J_0 \Rightarrow 43008 = 0.08\times J_0 + 0.82\times\left(64000-J_0\right) = J_0 = ...$ $\Rightarrow a = \frac{15360-\left(64000-J_0\right)}{J_0} = ...$	M1	3.1a
	$a = \frac{5683.2}{9472} = 0.60$	A1	1.1b
		(3)	
(iii)	Initial juvenile population = $\frac{"6144"}{"0.48"} = 12800$	M1	3.4
	So change of 2560 juvenile chimpanzees	A1	1.1b
		(2)	
(c)	As the number of juveniles has increased, the model is not initially predicting a decline, so is not suitable in the short term. (Follow through their answer to (b) – but they must have made an attempt at it to find at least a value for J_0)	B1ft	3.5a
		(1)	
(d)	Third category needs to be introduced for chimpanzees aged 40 and above, mature chimpanzees M_n, and a matrix multiplication of increased dimension set up. Accept $3\times3, 3\times2$ or 2×3 matrices including all three categories in the column vector.	M1	3.5c
	The corresponding matrix model will have the form $\begin{pmatrix} J_{n+1} \\ A_{n+1} \\ M_{n+1} \end{pmatrix} = \begin{pmatrix} a & b & \underline{\mathbf{0}} \\ 0.08 & c & 0 \\ 0 & d & e \end{pmatrix}\begin{pmatrix} J_n \\ A_n \\ M_n \end{pmatrix}$ (The underlined zero must be correct but do not be concerned about any values used in the other entries.)	A1	3.3
		(2)	
		(12 marks)	

Notes		
(a)	**B1**	Correct interpretation. Need not mention survival but must be clear it is the (proportion of) **juveniles that remain as juveniles** the next year (ie those that survive but don't progress to adulthood). E.g. accept "(number of) juveniles who do not become adults" but do not accept "surviving juveniles".
		Mark part (b) as a whole.
(b)(i)	**M1**	Attempts the determinant in terms of a Allow miscopies for the attempt. Allow $0.82a - 0.12$ as a slip.
	M1	Attempts the form of the inverse, swapped leading diagonals and sign changed on both off diagonals. Allow miscopies of the numbers but the signs must be correct.
	A1	Correct inverse matrix
(ii)	**M1**	Use the inverse matrix and attempts to find the initial juvenile and adult populations. . (May have determinant 1 for this mark.) Alternatively, sets up simultaneous equations from the original system, $15360 = aJ_0 + 0.15 \times A_0$ and $43008 = 0.08 \times J_0 + 0.82 \times A_0$ Accept with J_n and A_n or other appropriate variables.
	M1	Uses the sum of initial populations equals 64000 in an attempt to find a. (May have determinant 1 for this mark.) If using alternative, use of e.g. $A_0 = 64000 - J_0$ in second equation to find J_0, followed by attempt to find a. Award for an attempt to solve the equations, but don't be too concerned with the algebraic process as long as they are attempting to use all three equations.
	A1	Correct value, $a = 0.6$ (or 0.60 or $\frac{3}{5}$).
(iii)	**M1**	Uses their a to find the value of J_0. This mark may be gained for work done in (ii) if the alternative has been used but must have come from a correct method.
	A1	Correct difference found, as long as there is no contradictory statement – so "decrease of 2560" is A0.
(c)	**B1ft**	Comments that the change is an increase so does not fit the model. Follow through their answer to (b) as long as at least a value for J_0 has been found. If a decrease has been found allow for commenting the model is suitable. If an answer is given to (b)(iii), follow through on whatever their answer is. If no answer has been given, but an initial population found, a comparison should be made between this value and 153600 with conclusion must be consistent with their answer for J_0
(d)	**M1**	Introduces a third category (may be *M*ature, *E*lderly or any suitable letter used) and sets up a matrix multiplication (the left hand side may be missing for this mark) with all three categories in the column vector. The dimension of the matrix should be 3 in at least either row or column, and there should be a 3×1 vector.
	A1	Sets up the new matrix equation, including both sides and making clear the zero (underlined) so that the correct progression that no new juveniles arise from the mature chimpanzees is clear. Overlook other values, though ideally the other two zeroes are shown too, to indicate mature chimpanzees do not regress to adulthood, and juveniles cannot proceed directly to mature chimpanzees.

1. A system of three equations is defined by

$$\begin{aligned} kx + 3y - z &= 3 \\ 3x - y + z &= -k \\ -16x - ky - kz &= k \end{aligned}$$

where k is a positive constant.

Given that there is no unique solution to all three equations,

(a) show that $k = 2$

(2)

Using $k = 2$

(b) determine whether the three equations are consistent, justifying your answer.

(3)

(c) Interpret the answer to part (b) geometrically.

(1)

Question	Scheme	Marks	AOs
1(a)	$\begin{vmatrix} k & 3 & -1 \\ 3 & -1 & 1 \\ -16 & -k & -k \end{vmatrix} = k(k+k) - 3(-3k+16) - 1(-3k-16)$	M1	2.1
	Solves det = 0 $\Rightarrow 2k^2 + 12k - 32 = 0$ or $k^2 + 6k - 16 = 0$ To achieve k = 2 (k = – 8 must be rejected)	A1	1.1b
		(2)	
	Special case $\begin{vmatrix} 2 & 3 & -1 \\ 3 & -1 & 1 \\ -16 & -2 & -2 \end{vmatrix} = 2(2+2) - 3(-3\times 2+16) - 1(-3\times 2-16)$ Shows det = 0, therefore when k = 2 there is no unique solution	M1 A0	2.1 1.1b
(b)	Eliminates z to achieve **two** equations in x and y e.g. $5x + 2y = 1$ $-10x - 4y = -2$ $20x + 8y = 4$ Eliminates x to achieve **two** equations in y and z e.g. $11y - 5z = 13$ $22y - 10z = 26$ $-22y - 10z = -26$ Eliminates y to achieve **two** equations in x and z e.g. $11x + 2z = -3$ $22x + 4z = -6$ $-44x - 8z = 12$	M1 A1	3.1a 1.1b
	Must give a **reason:** e.g. Two equations are a linear multiple of each other e.g. shows they are the same equation therefore the equations are **consistent**.	A1	2.4
	Alternative Eliminates two different variables to form two equations, should be one equation from two of the three sections in the main scheme. e.g $5x + 2y = 1$ and $11y - 5z = 13$ rearranges and substitutes in to one of the original equations in three variables. e.g. $2x + 3\left(\frac{1-5x}{2}\right) - \left(\frac{-3-11x}{2}\right) = 3$	M1	3.1a
	Correct equations e.g $5x + 2y = 1$ and $11y - 5z = 13$	A1	1.1b
	Shows that the equations are a solution e.g. 3 = 3 therefore **consistent**	A1	2.4
(c)	The three **planes** form a **sheaf.**	B1	2.2a
		(1)	
			(6 marks)

Notes:
(a) **M1:** Finds the determinant of the matrix corresponding to the system of equations. **A1:** Sets determinant = 0 and solves their 3TQ to achieve $k = 2$ ($k = -8$ must be rejected)
(a) Special case **M1A0:** Uses $k = 2$ and finds the determinant of the matrix corresponding to the system of equations Shows that determinant = 0 and concludes that when $k = 2$ there is no unique solution
(b) **M1:** A complete method eliminating one variable from the equations using two different pairs of equations. Condone if a different value of k is used **A1:** Achieves two equations in the same two variables **A1: Must give a reason,** shows that the equations are a linear multiple of each other therefore they are **consistent**.
(b) Alternative **M1:** A complete method eliminating one variable from the equations using two different pairs of equations. Substitutes these equations into one of the original equations in three variables. **A1:** Achieves two correct equations in two different variables **A1:** Shows that the equation works therefore they are **consistent**.
(c) **B1:** The three **planes** form a **sheaf.** They must have full marks in (b) to award this mark.

6. (i)

$$\mathbf{A} = \begin{pmatrix} 2 & a \\ a-4 & b \end{pmatrix}$$

where a and b are non-zero constants.

Given that the matrix **A** is self-inverse,

(a) determine the value of b and the possible values for a.

(5)

The matrix **A** represents a linear transformation M.

Using the smaller value of a from part (a),

(b) show that the invariant points of the linear transformation M form a line, stating the equation of this line.

(3)

(ii)

$$\mathbf{P} = \begin{pmatrix} p & 2p \\ -1 & 3p \end{pmatrix}$$

where p is a positive constant.

The matrix **P** represents a linear transformation U.
The triangle T has vertices at the points with coordinates (1, 2), (3, 2) and (2, 5).
The area of the image of T under the linear transformation U is 15

(a) Determine the value of p.

(4)

The transformation V consists of a stretch scale factor 3 parallel to the x-axis with the y-axis invariant followed by a stretch scale factor –2 parallel to the y-axis with the x-axis invariant. The transformation V is represented by the matrix **Q**.

(b) Write down the matrix **Q**.

(2)

Given that U followed by V is the transformation W, which is represented by the matrix **R**,

(c) find the matrix **R**.

(2)

<table>
<tr><th>Question</th><th colspan="2">Scheme</th><th>Marks</th><th>AOs</th></tr>
<tr><td rowspan="7">6(i) (a)</td><td colspan="2">Multiplies the matrix A by itself and sets equal to I to form one equation in a only and another equation involving both a and b.
$\begin{pmatrix} 2 & a \\ a-4 & b \end{pmatrix}\begin{pmatrix} 2 & a \\ a-4 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow 4 + a(a-4) = 1$
and either $2a + ab = 0$ or $2(a-4) + b(a-4) = 0$ or $a(a-4) + b^2 = 1$</td><td>M1</td><td>3.1a</td></tr>
<tr><td colspan="2">Solves a 3TQ involving only the constant a. This could come after a value of b is found and this value substituted into an equation involving both a and b
$a^2 - 4a + 3 = 0 \Rightarrow (a-3)(a-1) = 0 \Rightarrow a = ...$</td><td>dM1</td><td>1.1b</td></tr>
<tr><td colspan="2">$a = 1, a = 3$</td><td>A1</td><td>11b</td></tr>
<tr><td>Substitutes a value for a into an equation involving both a and b and solves for b.
e.g.
$2(1) + (1)b \Rightarrow b = ...$
$2(1-4)b + (1-4) = 0 \Rightarrow b = ...$
$(1)(1-4) + b^2 = 1 \Rightarrow b = ...$</td><td>Alternatively uses
$2a + ab = 0$
$a(2+b) = 0$

As $a \neq 0 \;\; 2 + b = 0 \Rightarrow b = ...$</td><td>dM1</td><td>1.1b</td></tr>
<tr><td colspan="2">$b = -2$</td><td>A1</td><td>1.1b</td></tr>
<tr><td colspan="2"></td><td>(5)</td><td></td></tr>
</table>

<table>
<tr><td rowspan="3"></td><td colspan="2">Alternative (i) (a)
Finds $\mathbf{A}^{-1}$ in terms of a and b, sets equal to A and attempts to find at least two different equations. Allow a single sign slip
$\frac{1}{2b - a(a-4)}\begin{pmatrix} b & -a \\ -(a-4) & 2 \end{pmatrix} = \begin{pmatrix} 2 & a \\ a-4 & b \end{pmatrix}$

One equation from $\frac{b}{2b - a(a-4)} = 2$, $\frac{2}{2b - a(a-4)} = b$
One equation from $\frac{-a}{2b - a(a-4)} = a$, $\frac{-(a-4)}{2b - a(a-4)} = a - 4$</td><td>M1</td><td>3.1a</td></tr>
<tr><td>Uses their value of b and their value of the determinant to form and solve a 3TQ involving only the constant a
$a^2 - 4a + 3 = 0$
$\Rightarrow (a-3)(a-1) = 0$
$\Rightarrow a = ...$</td><td>Eliminates b from their equations and solve a 3TQ involving only the constant a
$a^2 - 4a + 3 = 0$
$\Rightarrow (a-3)(a-1) = 0$
$\Rightarrow a = ...$</td><td>dM1</td><td>1.1b</td></tr>
<tr><td colspan="2">$a = 1, a = 3$</td><td>A1</td><td>1.1b</td></tr>
</table>

	$\frac{-a}{2b-a(a-4)}=a$ $\Rightarrow 2b-a(a-4)=-1\Rightarrow \frac{b}{-1}=2$ Or $\frac{-(a-4)}{2b-a(a-4)}=a-4$ $\Rightarrow 2b-a(a-4)=-1$ $\Rightarrow \frac{2}{-1}=b$	Substitutes a value for *a* into an equation to find a value for *b*	dM1	1.1b
	$b=-2$		A1	1.1b
(b)	Uses their smallest value of **a** and their value for **b** to form two equations $\begin{pmatrix} 2 & 'a' \\ 'a-4' & 'b' \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}\Rightarrow 2x+ay=x$ and $(a-4)x+by=y$ $\begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}\Rightarrow 2x+y=x$ and $-3x-2y=y$		M1	3.1a
	$2x+y=x\Rightarrow x+y=0$ o.e. and $-3x-2y=y\Rightarrow x+y=0$ o.e.		M1	1.1b
	$x+y=0$ o.e.		A1	2.1
			(3)	
(ii)(a)	Area of the triangle T = 3		B1	1.1b
	Complete method to find a value for *p*. Need to see an attempt at the determinant and setting equal to 15 divided by their area of *T*. The resulting 3TQ needs to be solved to find a value of *p*. Determinant $3p\times p-(-1)\times 2p=\frac{15}{\text{'their area'}}\Rightarrow p=...$		M1	3.1a
	$3p^2+2p-5(=0)$		A1	1.1b
	$p=1$ must reject $p=-\frac{5}{3}$		A1	1.1b
			(4)	
(b)	$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$		B1 B1	1.1b 1.1b
			(2)	
(c)	(their matrix found in part (b)) $\begin{pmatrix} 'p' & 2'p' \\ -1 & 3'p' \end{pmatrix}=\begin{pmatrix} ... & ... \\ ... & ... \end{pmatrix}$ $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}=\begin{pmatrix} ... & ... \\ ... & ... \end{pmatrix}$		M1	1.1b

	$\begin{pmatrix} 3 & 6 \\ 2 & -6 \end{pmatrix}$	A1ft	1.1b
		(2)	
			(16 marks)

Notes:

(i)(a)

M1: Forming two equations, one involving *a* only and one involving *a* and *b*

dM1: Dependent on previous mark, solves a 3TQ involving *a*

A1: Correct values for *a*

dM1: Dependent on first method mark Substitutes one of their values of *a* into an equation involving *a* and *b* and solve to find a value for *b*. Alternatively factorises either $2a + ab = 0$ and uses $a \neq 0$ to find a value for *b*.

A1: Correct value for *b*

Alternative(i)(a)

M1: Finds $\mathbf{A}^{-1}$ and sets equal to **A** and forms two different equations

dM1: Dependent on previous mark. Eliminates *b* from their equations and solves a 3TQ involving only the constant *a*. Alternatively if the value of b is found first substitutes their value for *b* into their determinant = – 1 to form and solve a 3TQ for *a*

A1: Correct value for *a*

dM1: Dependent on first method mark. Substitutes a value for *a* into an equation to find a value for *b*. Alternatively uses one equation to find the determinant = – 1 and uses this to find a value of *b*.

A1: Correct values for *b*

(b)

M1: Extracts simultaneous equations using their matrix **A** with their smaller value of *a*.

M1: Gathers terms from their two equations.

A1: Achieves the correct equations and deduces the correct line. Accept equivalent equations as long as both have been shown to be the same.

(ii)(a)

B1: Area of the triangle $T = 3$

M1: Full method. Finds the determinant, sets equal to 15/their area and solves the resulting 3TQ

A1: Correct quadratic

A1: $p = 1$ only

(b)

B1 One correct row or column

B1: All correct

(c)

M1: Multiplies the matrices **QP** in the correct order (if answer only then evidence can be taken from 3 correct or 3 correct ft elements)

A1ft: Correct matrix following through on their answer to part (b) and their value of *p* as long as it is a positive constant

4.

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 4 \\ k & 2 & -2 \\ 4 & 1 & -2 \end{pmatrix} \qquad \mathbf{N} = \begin{pmatrix} k-7 & 6 & -10 \\ 2 & -20 & 24 \\ -3 & 2 & -1 \end{pmatrix}$$

where k is a constant.

(a) Determine, in simplest form in terms of k, the matrix $\mathbf{MN}$.

(2)

(b) Given that $k = 5$

(i) write down $\mathbf{MN}$

(ii) hence write down $\mathbf{M}^{-1}$

(2)

(c) Solve the simultaneous equations

$$\begin{aligned} 2x + y + 4z &= 2 \\ 5x + 2y - 2z &= 3 \\ 4x + y - 2z &= -1 \end{aligned}$$

(2)

(d) Interpret the answer to part (c) geometrically.

(1)

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Question	Scheme	Marks	AOs
4(a)	$\mathbf{MN}=\begin{pmatrix}2k-24 & 0 & 0\\ k^2-7k+10 & 6k-44 & -10k+50\\ 4k-20 & 0 & -14\end{pmatrix}$	B1 B1	1.1b 1.1b
		(2)	
(b)(i)	$\mathbf{MN}=\begin{pmatrix}-14 & 0 & 0\\ 0 & -14 & 0\\ 0 & 0 & -14\end{pmatrix}$	B1ft	1.1b
(ii)	$\mathbf{M}^{-1}=-\frac{1}{14}\begin{pmatrix}-2 & 6 & -10\\ 2 & -20 & 24\\ -3 & 2 & -1\end{pmatrix}$	B1	1.1b
		(2)	
(c)	$\mathbf{M}^{-1}=-\frac{1}{14}\begin{pmatrix}-2 & 6 & -10\\ 2 & -20 & 24\\ -3 & 2 & -1\end{pmatrix}\begin{pmatrix}2\\ 3\\ -1\end{pmatrix}=\ldots$	M1	1.1b
	$\left(-\frac{12}{7}, \frac{40}{7}, -\frac{1}{14}\right)$	A1	1.1b
		(2)	
(d)	The coordinates of the only point at which the **planes** represented by the equations in (c) meet.	B1	2.2a
		(1)	
			(7 marks)

Notes

(a)
B1: For 2 correct rows or 2 correct columns (allow unsimplified)
B1: Fully correct simplified matrix
(b)(i)
B1ft: Correct matrix (follow through from part (a)). If an error with part (a) allow the correct matrix stated, restart use of calculator.
(ii)
B1: Deduces the correct inverse matrix, may use calculator
(c)
M1: Any complete method to find the values of x, y and z (Must be using **their inverse** if using the method in the main scheme)
Allow use of a calculator
A1: Correct exact coordinates (allow as a vector or $x = \ldots, y = \ldots, z = \ldots$)
(d)
B1: Describes the correct geometrical configuration of the **planes**

1.

$$\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 7 & 2 \\ -5 & 8 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & 3 & 2 \\ -1 & 6 & 5 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} -5 & 2 & 1 \\ 4 & 3 & 8 \\ -6 & 11 & 2 \end{pmatrix}$$

Given that $\mathbf{I}$ is the 3×3 identity matrix,

(a) (i) show that there is an integer k for which

$$\mathbf{AB} - 3\mathbf{C} + k\mathbf{I} = \mathbf{0}$$

stating the value of k

(ii) explain why there can be no constant m such that

$$\mathbf{BA} - 3\mathbf{C} + m\mathbf{I} = \mathbf{0}$$

(4)

(b) (i) Show how the matrix $\mathbf{C}$ can be used to solve the simultaneous equations

$$\begin{aligned} -5x + 2y + z &= -14 \\ 4x + 3y + 8z &= 3 \\ -6x + 11y + 2z &= 7 \end{aligned}$$

(ii) Hence use your calculator to solve these equations.

(3)

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Question	Scheme	Marks	AOs
1(a)	(i) $\mathbf{AB}=\begin{pmatrix}4&-1\\7&2\\-5&8\end{pmatrix}\begin{pmatrix}2&3&2\\-1&6&5\end{pmatrix}=\begin{pmatrix}8+1&12-6&8-5\\14-2&21+12&14+10\\-10-8&-15+48&-10+40\end{pmatrix}=\begin{pmatrix}9&6&3\\12&33&24\\-18&33&30\end{pmatrix}$	M1	1.1b
	So $\mathbf{AB}-3\mathbf{C}=\begin{pmatrix}9&6&3\\12&33&24\\-18&33&30\end{pmatrix}-\begin{pmatrix}-15&6&3\\12&9&24\\-18&33&6\end{pmatrix}=\begin{pmatrix}24&0&0\\0&24&0\\0&0&24\end{pmatrix}$ or $\mathbf{AB}-3\mathbf{C}=\begin{pmatrix}9&6&3\\12&33&24\\-18&33&30\end{pmatrix}+\begin{pmatrix}15&-6&-3\\-12&-9&-24\\18&-33&-6\end{pmatrix}=\begin{pmatrix}24&0&0\\0&24&0\\0&0&24\end{pmatrix}$ and states a value for k	M1	1.1b
	Hence $\mathbf{AB}-3\mathbf{C}-24\mathbf{I}=\mathbf{0}$ so $k=-24$	A1	1.1b
	(ii) Need two things One of: • **BA** is a 2×2 matrix • Finds the matrix **BA** (must be a 2×2 matrix) AND One of: • cannot subtract a 3×3 matrix • finds matrix **3C** and comments that they have different dimensions / can't be done • can't subtract matrices of different sizes • 3**C** or **C** is a 3×3 matrix • **BA** needs to be a 3×3 matrix	B1	2.4
		(4)	
(b)(i)	$\begin{pmatrix}-5&2&1\\4&3&8\\-6&11&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-14\\3\\7\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-5&2&1\\4&3&8\\-6&11&2\end{pmatrix}^{-1}\begin{pmatrix}-14\\3\\7\end{pmatrix}$ Or states $\begin{pmatrix}x\\y\\z\end{pmatrix}=\mathbf{C}^{-1}\begin{pmatrix}-14\\3\\7\end{pmatrix}$ Or states $\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{360}\begin{pmatrix}-82&7&13\\-56&-4&44\\62&43&-23\end{pmatrix}\begin{pmatrix}-14\\3\\7\end{pmatrix}$	M1	1.2
(ii)	$=\frac{1}{360}\begin{pmatrix}-82&7&13\\-56&-4&44\\62&43&-23\end{pmatrix}\begin{pmatrix}-14\\3\\7\end{pmatrix}=\ldots$	M1	1.1b

	$= \begin{pmatrix} -\frac{41}{180} & \frac{7}{360} & \frac{13}{360} \\ -\frac{7}{45} & -\frac{1}{90} & \frac{11}{90} \\ \frac{31}{180} & \frac{43}{360} & -\frac{23}{360} \end{pmatrix} \begin{pmatrix} -14 \\ 3 \\ 7 \end{pmatrix} = \ldots$ $\mathbf{C}^{-1}\begin{pmatrix} -14 \\ 3 \\ 7 \end{pmatrix} = \ldots$		
	So solution is $x = \frac{7}{2}, y = 3, z = -\frac{5}{2}$ or $(3.5,\ 3,\ -2.5)$	A1	1.1b
		(3)	
			(7 marks)

Notes:

(a) (i)

M1: Attempts to find **AB**. Usually this will be done on calculator so answer implies the method. If answer is incorrect allow for at least 6 correct entries or calculations shown. This mark can be implied by a correct matrix for $\mathbf{AB} - 3\mathbf{C}$ gives the first M1

M1: Uses their **AB** and **3C** matrices to find a multiple **I** and states a value for k

A1: Correct proof with $k = -24$ seen explicitly (may be in equation).

Minimum working required is $\mathbf{AB} - 3\mathbf{C} = \begin{pmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{pmatrix}$ gets M1 then states a value for k M1 then $k = -24$ gets A1

Special case: If minimum working required is not seen and just $k = -24$ stated then M1 M0 A0 as they have not shown that the value of k works.

(ii)

B1: Correct explanation referring to the dimensions of **BA** and **C** (or 3**C**) and that they do not match in the equation. They can find both these matrices and then comment they cannot be subtracted.

(b) Mark (i) and (ii) altogether

M1: States or implies use of the correct method of using the inverse matrix.

M1: Carries out the process of multiplying after finding the inverse. May find inverse long hand first. Finding the inverse matrix then writes down an answer gains M1.

Note: There is no need to find the inverse matrix. If the inverse matrix is not stated just answers written down then two out of the three correct ordinates imply the M1.

A1: Correct solution. Must be clear that $x = \frac{7}{2}, y = 3, z = -\frac{5}{2}$ allow $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3.5 \\ 3 \\ -2.5 \end{pmatrix}$

Note: If they solve using simultaneous equations only this is M0 M0 A0

If there is no reference to the inverse matrix and correct answers stated this is M0 M0 A0