# Cp1Ch4 XMQs and MS

(Total: 144 marks)

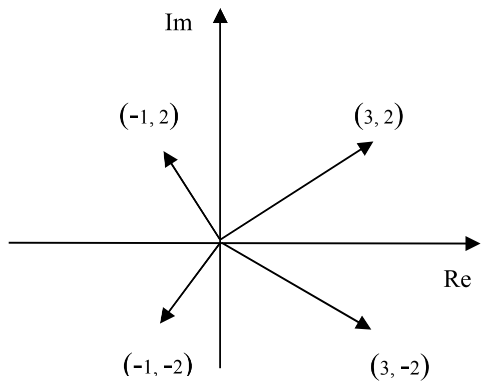
1. CP1_Sample Q3 . 9 marks - CP1ch4 Roots of polynomials
2. CP2_Sample Q1 . 8 marks - CP1ch4 Roots of polynomials
3. CP1_Specimen Q6 . 9 marks - CP1ch4 Roots of polynomials
4. CP1_2019 Q1 . 9 marks - CP1ch1 Complex numbers
5. CP2_2019 Q2 . 8 marks - CP1ch4 Roots of polynomials
6. CP1_2020 Q1 . 10 marks - CP1ch4 Roots of polynomials
7. CP1_2021 Q3 . 6 marks - CP1ch4 Roots of polynomials
8. CP1_2022 Q1 . 6 marks - CP1ch4 Roots of polynomials
9. CP2_2022 Q6 . 10 marks - CP1ch4 Roots of polynomials
10. CP(AS)_2018 Q2 . 5 marks - CP1ch4 Roots of polynomials
11. CP(AS)_2018 Q7 . 7 marks - CP1ch4 Roots of polynomials
12. CP(AS)_2019 Q2 . 5 marks - CP1ch4 Roots of polynomials
13. CP(AS)_2019 Q5 . 9 marks - CP1ch2 Argand diagrams
14. CP(AS)_2019 Q7 . 8 marks - CP1ch4 Roots of polynomials
15. CP(AS)_2020 Q7 . 6 marks - CP1ch4 Roots of polynomials
16. CP(AS)_2020 Q9 . 6 marks - CP1ch4 Roots of polynomials
17. CP(AS)_2021 Q2 . 5 marks - CP1ch4 Roots of polynomials
18. CP(AS)_2021 Q7 . 9 marks - CP1ch4 Roots of polynomials
19. CP(AS)_2022 Q4 . 9 marks - CP1ch4 Roots of polynomials

**3.** $$f(z) = z^4 + az^3 + 6z^2 + bz + 65$$

where $a$ and $b$ are real constants.

Given that $z = 3 + 2\mathbf{i}$ is a root of the equation $f(z) = 0$, show the roots of $f(z) = 0$ on a single Argand diagram.

**(9)**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **3** | $z = 3 - 2\text{i}$ is also a root | B1 | 1.2 |
| | $\left(z-(3+2\text{i})\right)\left(z-(3-2\text{i})\right)=\ldots$ <br> **or** <br> Sum of roots = 6, Product of roots = 13 $\Rightarrow \ldots$ | M1 | 3.1a |
| | $= z^2 - 6z + 13$ | A1 | 1.1b |
| | $\left(z^4 + az^3 + 6z^2 + bz + 65\right) = \left(z^2 - 6z + 13\right)\left(z^2 + cz + 5\right) \Rightarrow c = \ldots$ | M1 | 3.1a |
| | $z^2 + 2z + 5 = 0$ | A1 | 1.1b |
| | $z^2 + 2z + 5 = 0 \Rightarrow z = \ldots$ | M1 | 1.1a |
| | $z = -1 \pm 2\text{i}$ | A1 | 1.1b |
| |   | B1 <br> $3 \pm 2\text{i}$ <br> Plotted correctly | 1.1b |
| | | B1ft <br> $-1 \pm 2\text{i}$ <br> Plotted correctly | 1.1b |
| | | | **(9 marks)** |

**Notes:**

**B1:** Identifies the complex conjugate as another root
**M1:** Uses the conjugate pair and a correct method to find a quadratic factor
**A1:** Correct quadratic
**M1:** Uses the given quartic and their quadratic to identify the value of *c*
**A1:** Correct 3TQ
**M1:** Solves their second quadratic
**A1:** Correct second conjugate pair
**B1:** First conjugate pair plotted correctly and labelled
**B1ft:** Second conjugate pair plotted correctly and labelled (Follow through their second conjugate pair)

**Answer ALL questions. Write your answers in the spaces provided.**

**1.** The roots of the equation

$$x^3 - 8x^2 + 28x - 32 = 0$$

are $\alpha$, $\beta$ and $\gamma$

Without solving the equation, find the value of

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

(ii) $(\alpha + 2)(\beta + 2)(\gamma + 2)$

(iii) $\alpha^2 + \beta^2 + \gamma^2$

**(8)**

## Paper 2: Core Pure Mathematics 2 Mark Scheme

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| 1(i) | $\alpha+\beta+\gamma=8,\quad \alpha\beta+\beta\gamma+\gamma\alpha=28,\quad \alpha\beta\gamma=32$ | B1 | 3.1a |
| | $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$ | M1 | 1.1b |
| | $=\frac{7}{8}$ | A1ft | 1.1b |
| | | **(3)** | |
| (ii) | $(\alpha+2)(\beta+2)(\gamma+2)=(\alpha\beta+2\alpha+2\beta+4)(\gamma+2)$ | M1 | 1.1b |
| | $=\alpha\beta\gamma+2(\alpha\beta+\alpha\gamma+\beta\gamma)+4(\alpha+\beta+\gamma)+8$ | A1 | 1.1b |
| | $=32+2(28)+4(8)+8=128$ | A1 | 1.1b |
| | | **(3)** | |
| | **Alternative:** | | |
| | $(x-2)^3-8(x-2)^2+28(x-2)-32=0$ | M1 | 1.1b |
| | $=\ldots-8+\ldots-32+\ldots-56-32=-128$ | A1 | 1.1b |
| | $\therefore(\alpha+2)(\beta+2)(\gamma+2)=128$ | A1 | 1.1b |
| | | **(3)** | |
| (iii) | $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$ | M1 | 3.1a |
| | $=8^2-2(28)=8$ | A1ft | 1.1b |
| | | **(2)** | |
| | | | **(8 marks)** |

**Notes:**

**(i)**

**B1:** Identifies the correct values for all 3 expressions (can score anywhere)

**M1:** Uses a correct identity

**A1ft:** Correct value (follow through their 8, 28 and 32)

**(ii)**

**M1:** Attempts to expand

**A1:** Correct expansion

**A1:** Correct value

**Alternative:**

**M1:** Substitutes $x-2$ for $x$ in the given cubic

**A1:** Calculates the correct constant term

**A1:** Changes sign and so obtains the correct value

**(iii)**

**M1:** Establishes the correct identity

**A1ft:** Correct value (follow through their 8, 28 and 32)

**6.**

$$f(x) = kx^2 + 3x - 11 \qquad g(x) = mx^3 - 2x^2 + 3x - 9$$

where $k$ and $m$ are real constants.

Given that

- the sum of the roots of f is equal to the product of the roots of g
- g has at least one root on the imaginary axis

(a) solve completely

(i) $f(x) = 0$

(ii) $g(x) = 0$

**(7)**

(b) Plot the roots of f and the roots of g on a single Argand diagram.

**(2)**

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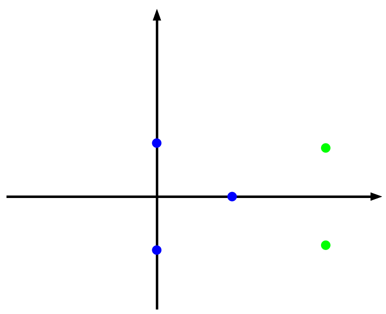
| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **6(a)** | Attempts sum of roots of f (= $-3/k$) and product of roots of g (= $9/m$) and uses them to form a relationship between $k$ and $m$. | M1 | 3.1a |
| | So $-3/k = 9/m$ | A1 | 1.1b |
| | Sum of roots of g is $2/m \Rightarrow 2/m$ is a root of g as the other roots have no real part.<br>OR root on imaginary axis has form $\alpha\text{i}$, and substituting in g and equating real and imaginary terms gives $2\alpha^2 - 9 = 0$ & $3\alpha - m\alpha^3 = 0$ | B1 | 3.1a |
| | $g(2/m) = 0 \Rightarrow m(2/m)^3 - 2(2/m)^2 + 3(2/m) - 9 = 0 \Rightarrow m = \ldots \quad (m = 2/3)$<br>OR $\alpha^2 = \frac{9}{2} \neq 0 \Rightarrow m = \frac{3}{\alpha^2} = \ldots\left(= \frac{2}{3}\right)$ | M1 | 1.1a |
| | So $g(x) = 0 \Rightarrow \left(\frac{2}{3}(x-3)\left(x^2 + \frac{9}{2}\right) = 0 \Rightarrow\right) \; x = 3, \pm\frac{3\sqrt{2}}{2}\text{i}$ | M1 | 1.1b |
| | $k = -2/9$, $f(x) = 0 \Rightarrow x = \frac{-3 \pm \sqrt{3^2 - 4(-2/9)(-11)}}{2(-2/9)} = \ldots$ | M1 | 2.2a |
| | $x = 3, \pm\frac{3\sqrt{2}}{2}\text{i}, \frac{27}{4} \pm \frac{3\sqrt{7}}{4}\text{i}$ | A1 | 1.1b |
| | | **(7)** | |
| **(b)** |  Correct roots for f plotted (shown green).<br>Correct roots for g plotted (shown blue). | B1ft<br>B1ft | 1.1b<br>1.1b |
| | | **(2)** | |
| | | | **(9 marks)** |

**Notes:**

**(a)**

**M1:** Identifies sum of roots of f or product of roots of g correctly

**A1:** Correct equation between $k$ and $m$

**B1:** Realises that g having roots on the imaginary axis means the sum of roots is equal to the only real root of the equation or forms correct simultaneous equations after substituting $\alpha\text{i}$ into g

**M1:** Uses factor theorem with their real root to find $m$ or solves their equations to find $m$

**M1:** Uses their $m$ to solve g($x$). May just see answers from calculator, or can factorise or complete the square.

**M1:** Deduces the correct value for $k$ and solves f($x$) using it. May just see answers from calculator, or can factorise or complete the square.

| |
|---|
| **A1**: All five roots correct – may not all be listed in one line, as long as the roots of g and f are clear. Accept exact equivalents. |
| **(b)**<br>**B1ft:** Correct roots for f plotted, follow through as long as they are complex. If answers to (b) are correct these should be further from the imaginary axis than the real root of g.<br>**B1ft:** Correct roots for g plotted, follow through their roots as long as two are on the imaginary axis. |

**Answer ALL questions. Write your answers in the spaces provided.**

**1.**

$$f(z) = z^4 + az^3 + bz^2 + cz + d$$

where $a$, $b$, $c$ and $d$ are real constants.

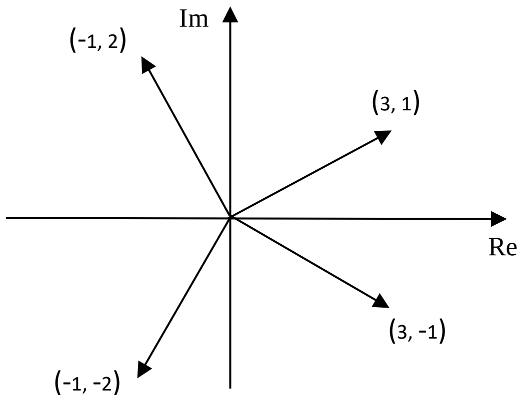
Given that $-1 + 2i$ and $3 - i$ are two roots of the equation $f(z) = 0$

(a) show all the roots of $f(z) = 0$ on a single Argand diagram,

**(4)**

(b) find the values of $a$, $b$, $c$ and $d$.

**(5)**

<table>
<tr><th>Question</th><th colspan="2">Scheme</th><th>Marks</th><th>AOs</th></tr>
<tr><td rowspan="5">1(a)</td><td colspan="2">$z=-1-2\mathrm{i}$ or $z=3+\mathrm{i}$</td><td>M1</td><td>1.2</td></tr>
<tr><td colspan="2">$z=-1-2\mathrm{i}$ and $z=3+\mathrm{i}$</td><td>A1</td><td>1.1b</td></tr>
<tr><td colspan="2" rowspan="2">
</td><td>B1</td><td>1.1b</td></tr>
<tr><td>B1</td><td>1.1b</td></tr>
<tr><td colspan="2"></td><td>(4)</td><td></td></tr>
<tr><td rowspan="6">(b)<br>Way 1</td><td>$(z-(-1+2\mathrm{i}))(z-(-1-2\mathrm{i}))=\ldots$<br>or<br>$(z-(3+\mathrm{i}))(z-(3-\mathrm{i}))=\ldots$</td><td>$\mathrm{f}(z)=(z-(-1+2\mathrm{i}))(z-(-1-2\mathrm{i}))$<br>$(z-(3+\mathrm{i}))(z-(3-\mathrm{i}))=\ldots$</td><td>M1</td><td>3.1a</td></tr>
<tr><td>$z^2+2z+5$ or $z^2-6z+10$</td><td>e.g. $\mathrm{f}(z)=(z^2+2z+5)(\ldots)$</td><td>A1</td><td>1.1b</td></tr>
<tr><td>$z^2+2z+5$ and $z^2-6z+10$</td><td>$\mathrm{f}(z)=(z^3+z^2(-1-\mathrm{i})+z(-1+2\mathrm{i})-15-5\mathrm{i})(\ldots)$</td><td>A1</td><td>1.1b</td></tr>
<tr><td>$\mathrm{f}(z)=(z^2+2z+5)(z^2-6z+10)$</td><td>Expands the brackets to forms a quartic</td><td>M1</td><td>3.1a</td></tr>
<tr><td colspan="2">$\mathrm{f}(z)=z^4-4z^3+3z^2-10z+50$ or<br>States $a=-4, b=3, c=-10, d=50$</td><td>A1</td><td>1.1b</td></tr>
<tr><td colspan="2"></td><td>(5)</td><td></td></tr>
</table>

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **Way 2** | $\text{sum roots} = \alpha+\beta+\gamma+\delta = (-1+2i)+(-1-2i)+(3+i)+(3-i) = \ldots$<br>$\text{pair sum} = \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta$<br>$=(-1+2i)(-1-2i)+(-1+2i)(3-i)+(-1+2i)(3+i)+(-1-2i)(3-i)$<br>$+(-1-2i)(3+i)+(3+i)(3-i)=\ldots$<br>$\text{triple sum} = \alpha\beta\gamma+\alpha\beta\delta+\beta\gamma\delta+\alpha\gamma\delta$<br>$=(-1+2i)(-1-2i)(3-i)+(-1+2i)(-1-2i)(3+i)+(-1+2i)(3+i)(3-i)$<br>$+(-1-2i)(3+i)(3-i)=\ldots$<br>$\text{Product} = \alpha\beta\gamma\delta = (-1+2i)(-1-2i)(3-i)(3+i)=\ldots$ | M1 | 3.1a |
| | sum = 4, pair sum = 3, triple sum = 10 and product = 50 | A1<br>A1 | 1.1b<br>1.1b |
| | $a = -(\text{their sum roots}) = -4$<br>$b = +(\text{their pair sum}) = 3$<br>$c = -(\text{triple sum}) = -10$<br>$d = +(\text{product}) = 50$ | M1<br>A1 | 3.1a<br>1.1b |
| | | **(5)** | |
| **Way 3** | $f(z) = (-1+2i)^4 + a(-1+2i)^3 + b(-1+2i)^2 + c(-1+2i) + d = 0$<br>$f(z) = (3+i)^4 + a(3+i)^3 + b(3+i)^2 + c(3+i) + d = 0$ | M1 | 3.1a |
| | Leading to<br>$-7+11a-3b-c+d=0 \quad 24-2a-4b+2c=0$<br>$28+18a+8b+3c+d=0 \quad 96+26a+6b+c=0$ | A1<br>A1 | 1.1b<br>1.1b |
| | Solves their simultaneous equation to find a value for one of the constants | M1 | 3.1a |
| | $a=-4, b=3, c=-10, d=50$ | A1 | 1.1b |
| | | **(5)** | |
| | | | **(9 marks)** |

**Notes**

(a)

M1: Identifies at least one correct complex conjugate as another root (can be seen/implied by Argand diagram)

A1: Both complex conjugate roots identified correctly (can be seen/implied by Argand diagram)

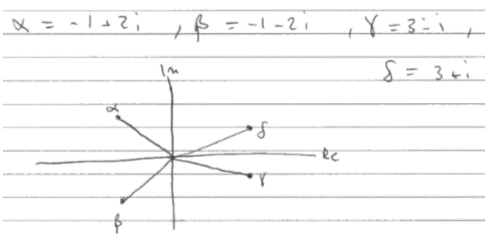
For the next two marks allow either a cross, dot or line drawn where the end point is labelled with the correct coordinate, corresponding complex number or clearly plotted with correct numbers labelled on the axis or indication of the correct coordinates by use of scale markers. Condone (3, i) etc. The axes do not need to be labelled with Re and Im.

B1: One complex conjugate pair correctly plotted.

B1: Both complex conjugate pair correctly plotted. The $3 \pm \mathrm{i}$ must be closer to the real axes than the $-1 \pm 2\mathrm{i}$

**If there is no indication of the coordinates, scale or complex numbers on the Argand diagram this is B0 B0.**

**Do accept correct labelling e.g.**




(b)

**Way 1**

M1: Correct strategy for forming at least one of the quadratic factors. Follow through their roots.

A1: At least one correct simplified quadratic factor.

A1: Both simplified quadratic factors correct or a correct simplified cubic factor

M1: A complete strategy to find values for *a*, *b*, *c* and *d* e.g. uses their quadratic factors or cubic and linear factor to form a quartic.

A1: Correct quartic in terms of *z* or correct values for *a*, *b*, *c* and *d* stated.

**Way 2**

M1: Correct strategy for finding at least three of the sum roots, pair sum, triple sum and product. Follow through their roots. This can be implied by at least three correct values for the sum roots, pair sum, triple sum and product with no working shown. If the calculations are not shown for the sums and product and they have at least two incorrect values this is M0.

A1: At least two correct values for the sum roots, pair sum, triple sum or product.

A1: All correct values for the sum, pair sum, triple sum and product.

M1: Must have real values of *a*, *b* , *c* and *d* and use *a* = −their sum roots, *b* = their pair sum, *c* = −their triple sum and *d* = their product.

A1: Correct quartic in terms of *z* or correct values for *a*, *b*, *c* and *d* stated.

**Way 3**

M1: Substitutes two roots into $\mathrm{f}(z) = 0$ and equates coefficients to form 4 equations

A1: At least two correct equations.

A1: All four correct equations

| |
|---|
| M1: Solve their four equation (using calculator) to find at least one value. This will need checking if incorrect equations used.<br>A1: Correct quartic in terms of *z* or correct values for *a*, *b*, *c* and *d* stated.<br><br>**Note:** Correct answer only will score 5/5 |

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| 2 | $\dfrac{8x-12}{(2x^2+3)(x+1)} = \dfrac{Ax+B}{2x^2+3}+\dfrac{C}{x+1}$ | M1 | 3.1a |
| | $8x-12=(Ax+B)(x+1)+C(2x^2+3)$<br>E.g. $x=-1\Rightarrow C=-4, x=0\Rightarrow B=0, x=1\Rightarrow A=8$<br>Or<br>Compares coefficients and solves<br>$(A+2C=0 \quad A+B=8 \quad B+3C=-12)$<br>$\Rightarrow A=..., B=..., C=...$ | dM1 | 1.1b |
| | $A=8 \quad B=0 \quad C=-4$ | A1 | 1.1b |
| | $\int\left(\dfrac{8x}{2x^2+3}-\dfrac{4}{x+1}\right)\mathrm{d}x = 2\ln(2x^2+3)-4\ln(x+1)$ | A1ft | 1.1b |
| | $2\ln(2x^2+3)-4\ln(x+1)=\ln\left(\dfrac{(2x^2+3)^2}{(x+1)^4}\right)$<br>or<br>$2\ln(2x^2+3)-4\ln(x+1)=2\ln\left(\dfrac{(2x^2+3)}{(x+1)^2}\right)$ | M1 | 2.1 |
| | $\lim\limits_{x\to\infty}\left\{\ln\dfrac{(2x^2+3)^2}{(x+1)^4}\right\}=\ln 4$ or $\lim\limits_{x\to\infty}\left\{2\ln\dfrac{(2x^2+3)}{(x+1)^2}\right\}=2\ln 2$ | B1 | 2.2a |
| | $\Rightarrow \int_0^\infty \dfrac{8x-12}{(2x^2+3)(x+1)}\mathrm{d}x=\ln\dfrac{4}{9}$ cao | A1 | 1.1b |
| | | **(7)** | |
| | | **(7 marks)** | |

**Notes**

M1: Selects the correct form for partial fractions.
dM1: Full method for finding values for all three constants. Dependent on having the correct form for the partial fractions. Allow slips as long as the intention is clear.
A1: Correct constants or partial fractions.

**2.** The roots of the equation

$$x^3 - 2x^2 + 4x - 5 = 0$$

are $p$, $q$ and $r$.

Without solving the equation, find the value of

(i) $\dfrac{2}{p} + \dfrac{2}{q} + \dfrac{2}{r}$

(ii) $(p - 4)(q - 4)(r - 4)$

(iii) $p^3 + q^3 + r^3$

**(8)**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| 2(i) | $p+q+r=2, \quad pq+pr+qr=4, \quad pqr=5$ | B1 | 3.1a |
| | $\frac{2}{p}+\frac{2}{q}+\frac{2}{r}=\frac{2(pq+pr+qr)}{pqr}$ | M1 | 1.1b |
| | $=\frac{8}{5}$ | A1ft | 1.1b |
| | | **(3)** | |
| | **Alternative for part (i)** | | |
| | $x=\frac{2}{y}\Rightarrow\frac{8}{y^3}-\frac{8}{y^2}+\frac{8}{y}-5=0\Rightarrow 5y^3-8y^2+8y-8=0$ | B1 | 3.1a |
| | $\frac{2}{p}+\frac{2}{q}+\frac{2}{r}=-\frac{-8}{5}$ | M1 | 1.1b |
| | $=\frac{8}{5}$ | A1ft | 1.1b |
| | | **(3)** | |
| (ii) | $(p-4)(q-4)(r-4)=(pq-4p-4q+16)(r-4)$<br>$=pqr-4pq-4pr-4qr+16p+16q+16r-64$ | M1<br>A1 | 1.1b<br>1.1b |
| | $\left(=pqr-4(pq+pr+qr)+16(p+q+r)-64\right)$ | | |
| | $=5-4(4)+16(2)-64=-43$ | A1 | 1.1b |
| | | **(3)** | |
| | **Alternative for part (ii)** | | |
| | $(x+4)^3-2(x+4)^2+4(x+4)-5=0$ | M1 | 1.1b |
| | $=...64+...-32+...16+...-5=43$ | A1 | 1.1b |
| | $\therefore(p-4)(q-4)(r-4)=-43$ | A1 | 1.1b |
| | | **(3)** | |
| (iii) | E.g.<br>$p^3+q^3+r^3=$<br>$=(p+q+r)^3-3(p+q+r)(pq+pr+qr)+3pqr$<br>or<br>$=(p+q+r)\left((p+q+r)^2-2(pq+pr+qr)-pq-pr-qr\right)+3pqr$<br>or<br>$=2\left((p+q+r)^2-2(pq+pr+qr)\right)-4(p+q+r)+3pqr$<br>$\Rightarrow p^3+q^3+r^3=...$ | M1 | 3.1a |
| | $=2^3-3(2)(4)+3(5)=-1$<br>$=2\left(2^2-3(4)\right)+3(5)=-1$<br>$=2\left(2^2-2(4)\right)-4(2)+3(5)=-1$ | A1 | 1.1b |
| | | **(2)** | |

| | Alternative for part (iii) | | |
|---|---|---|---|
| | $p^3-2p^2+4p-5=0,\ q^3-2q^2+4q-5=0,\ r^3-2r^2+4r-5=0$<br>$p^3+q^3+r^3-2\left(p^2+q^2+r^2\right)+4\left(p+q+r\right)-15=0$<br>$p^3+q^3+r^3=2\left(\left(p+q+r\right)^2-2\left(pq+pr+qr\right)\right)-4\left(p+q+r\right)+15$<br>$\Rightarrow p^3+q^3+r^3=...$ | M1 | 3.1a |
| | $=2\left(2^2-2\left(4\right)\right)-4\left(2\right)+15=-1$ | A1 | 1.1b |
| | | **(2)** | |
| | | | **(8 marks)** |

**Notes**

(i)
B1: Identifies the correct values for all 3 expressions (can score anywhere). Allow notation such as $\sum p,\ \sum pq$ for the sum and pair sum.
M1: Uses a correct identity for the sum
A1ft: Correct value (follow through their 2, 4 and 5)
**Alternative:**
B1: Obtains the correct cubic in "*y*"
M1: Uses a correct method
A1ft: Correct value (follow through their 2, 4 and 5)
(ii)
M1: Attempt to expand – must have an expression that involves the sum, pair sum and product
A1: Correct expansion
A1: Correct value
**Alternative:**
M1: Substitutes $x + 4$ for $x$ in the given cubic
A1: Calculates the correct constant term
A1: Correct value
(iii)
M1: Establishes a correct identity that is in terms of the sum, pair sum and product and substitutes to reach a numerical expression for $p^3+q^3+r^3$
A1: Correct value

**1.** $$f(z) = 3z^3 + pz^2 + 57z + q$$

where $p$ and $q$ are real constants.

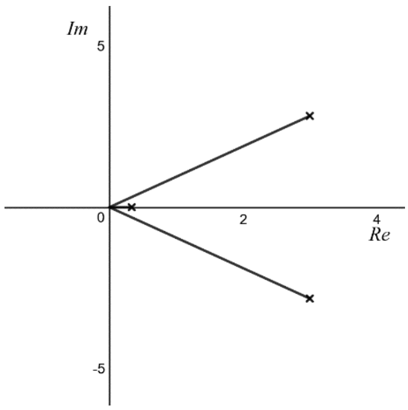
Given that $3 - 2\sqrt{2}\text{i}$ is a root of the equation $f(z) = 0$

(a) show all the roots of $f(z) = 0$ on a single Argand diagram,

**(7)**

(b) find the value of $p$ and the value of $q$.

**(3)**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **1(a)** | $\beta = 3+2\sqrt{2}\text{i}$ is also a root | B1 | 1.2 |
| | $\alpha\beta = 17, \alpha+\beta = 6$ | B1 | 1.1b |
| | $\alpha\beta+\alpha\gamma+\beta\gamma = \dfrac{57}{3}$ | M1 | 1.1b |
| | $\alpha\gamma+\beta\gamma = \dfrac{57}{3} - 17 = \gamma\left(\alpha+\beta\right) = 6\gamma \Rightarrow \gamma = ...$ | M1 | 3.1a |
| | $\gamma = \dfrac{1}{3}$ | A1 | 2.2a |
| |  | B1<br>B1ft | 1.1b<br>1.1b |
| | | **(7)** | |
| | **(a) Alternative:** | | |
| | $\beta = 3+2\sqrt{2}\text{i}$ is also a root | B1 | 1.2 |
| | $\left(z-\left(3+2\sqrt{2}\text{i}\right)\right)\left(z-\left(3-2\sqrt{2}\text{i}\right)\right) = z^2-6z+17$ | B1 | 1.1b |
| | $\text{f}\left(z\right) = \left(z^2-6z+17\right)\left(3z+a\right) = 3z^3+az^2-18z^2-6az+51z+17a$ | M1 | 1.1b |
| | $\Rightarrow 51-6a = 57 \Rightarrow a = -1 \Rightarrow \gamma = ...$ | M1 | 3.1a |
| | $\gamma = \dfrac{1}{3}$ | A1 | 2.2a |
| | Then B1 B1ft as above | | |
| | | **(7)** | |
| **(b)** | $3-2\sqrt{2}\text{i}+3+2\sqrt{2}\text{i}+\dfrac{1}{3} = -\dfrac{p}{3} \Rightarrow p = ...$<br>or<br>$\left(3-2\sqrt{2}\text{i}\right)\left(3+2\sqrt{2}\text{i}\right)\times\dfrac{1}{3} = -\dfrac{q}{3} \Rightarrow q = ...$ | M1 | 3.1a |
| | $p = -19$ **or** $q = -17$ | A1 | 1.1b |
| | $p = -19$ **and** $q = -17$ | A1 | 1.1b |
| | | **(3)** | |
| | **(b) Alternative:** | | |
| | $\text{f}\left(z\right) = \left(z^2-6z+17\right)\left(3z-1\right) = 3z^3+pz^2+57z+q$ | M1 | 3.1a |

| | | | |
|---|---|---|---|
| | $\Rightarrow p = ..., q = ...$ | | |
| | $p = -19$ **or** $q = -17$ | A1 | 1.1b |
| | $p = -19$ **and** $q = -17$ | A1 | 1.1b |
| | | **(3)** | |
| | | | **(10 marks)** |

**Notes**

(a)

B1: Identifies the correct complex conjugate as another root

B1: Correct values for the sum and product for the conjugate pair

M1: Correct application of the pair sum

M1: Identifies a complete and correct strategy for identifying the third root

A1: Deduces the correct third root

B1: $3 \pm 2\sqrt{2}\,\mathrm{i}$ plotted correctly, in quadrants 1 and 4 which are reflections in the real axis. Do not be concerned about labelling or scaling.

B1ft: Their real root plotted correctly, in correct relative position to the two complex roots. Scales are not needed but if correct, the real root must be close to the origin compared to the complex roots.

**Alternative:**

B1: Identifies the correct complex conjugate as another root

B1: Correct quadratic factor obtained

M1: Expands their quadratic×(3*z* + "*a*") or attempts to factor out the quadratic, or use long division, leading to a factor (3*z* + "*a*"). Implied by seeing $(z^2 - 6z + 17)(3z + a)$ with any value of *a* (or with their quadratic).

M1: Proceeds to extract the root from their third factor of from (3*z* + "*a*").

A1: Deduces the correct third root. If not explicitly stated, look for it on their diagram.

B1: $3 \pm 2\sqrt{2}\,\mathrm{i}$ plotted correctly, as above

B1ft: Their real root plotted correctly as above.

(b)

M1: Correct strategy used for identifying at least one of *p* or *q*

A1: At least one value correct

A1: Both values correct

**Alternative:**

M1: Correct strategy by expanding their quadratic and linear factors to identifying at least one of *p* or *q*

A1: At least one value correct

A1: Both values correct

Note: some may attempt to use the factor theorem with the complex root.

$\mathrm{f}\left(3 - 2\mathrm{i}\sqrt{2}\right) = 36 + p + q + \mathrm{i}\left(-228\sqrt{2} - 12\sqrt{2}p\right) = 0$

2nd B1: equate real and imaginary components to 0 to get correct equations

$36 + p + q = 0, -228\sqrt{2} - 12\sqrt{2}p = 0$

1st M1: solves their equations $\Rightarrow p = -19, q = -17$

2nd M1: Solves the cubic (may be from calculator). The 1st B1 may then be implied for the second complex root, and the rest as main scheme.

**3.** The cubic equation

$$ax^3 + bx^2 - 19x - b = 0$$

where $a$ and $b$ are constants, has roots $\alpha$, $\beta$ and $\gamma$

The cubic equation

$$w^3 - 9w^2 - 97w + c = 0$$

where $c$ is a constant, has roots $(4\alpha - 1)$, $(4\beta - 1)$ and $(4\gamma - 1)$

Without solving either cubic equation, determine the value of $a$, the value of $b$ and the value of $c$.

**(6)**





P 6 6 7 9 6 A 0 1 0 3 6

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **3** | $w=4x-1 \Rightarrow x=\frac{w+1}{4}$ | B1 | 3.1a |
| | $a\left(\frac{w+1}{4}\right)^3+b\left(\frac{w+1}{4}\right)^2-19\left(\frac{w+1}{4}\right)-b\ (=0)$ or<br>$(4x-1)^3-9(4x-1)^2-97(4x-1)+c\ (=0)$ | M1 | 3.1a |
| | $aw^3+(3a+4b)w^2+(3a+8b-304)w+(a-60b-304)=0$<br>or $64x^3-192x^2-304x+87+c=0$ | M1 | 1.1b |
| | Divides by $a$ and equates the coefficients of $w^2$ and $w$<br>$\frac{3a+4b}{a}=-9 \quad \frac{3a+8b-304}{a}=-97$<br>and solves simultaneously to find a value for $a$ or a value for $b$<br>**<u>Note:</u>** $12a+4b=0$ and $100a+8b=304$<br>or<br>Divides through by '16' leading to values of $a$ and $b$<br>$4x^3-12x^2-19x+\frac{87+c}{19}=0$ | M1 | 3.1a |
| | $c=\frac{a-60b-304}{a}=\ldots$ or $\frac{87+c}{19}=12 \Rightarrow c=\ldots$ | M1 | 1.1b |
| | $a=4 \quad b=-12 \quad c=105$ | A1 | 1.1b |
| | | **(6)** | |
| | | | **(6 marks)** |

**Notes:**

**B1:** Selects the method of making a connection between $x$ and $w$ by writing $w=4x-1$ or $x=\frac{w+1}{4}$

**M1:** Applies the process of substituting their $x=\frac{w+1}{4}$ into $ax^3+bx^2-19x-b=0$ or $w=4x-1$ into $w^3-9w^2-97w+c=0$. Must be substitution of the correct variable into the opposing equation but may be scored if the initial linear equation is incorrect (e.g. $x=4w-1$ into the first equation). Note that the " = 0 " can be missing for this mark.

**M1:** Expands the brackets and collects terms in their equation (in $x$ or $w$). Note that the " = 0 " can be missing for this mark.

**M1:** A complete method for finding a value for $a$ or $b$. See scheme, it involves dividing through by an appropriate factor for their equation to balance the $w^3$ or $-19x$ terms, then equating other coefficients and solving equations if necessary.

**M1:** A complete method for finding a value for $c$. They must have divided through by an appropriate factor as per the previous M before attempting to compare the constant coefficient (and use their $a$ and $b$ if appropriate).

**A1:** $a=4 \quad b=-12 \quad c=105$

| | | | |
|---|---|---|---|
| | | | |
| | **Alternative**<br>At least two of $\alpha+\beta+\gamma=-\frac{b}{a}$ $\alpha\beta+\alpha\gamma+\beta\gamma=-\frac{19}{a}$ $\alpha\beta\gamma=\frac{b}{a}$ | B1 | 3.1a |
| | New sum $=4(\alpha+\beta+\gamma)-3=9 \Rightarrow 4\left(-\frac{b}{a}\right)-3=9 \Rightarrow b=-3a$ | M1 | 3.1a |
| | New pair sum=$16(\alpha\beta+\alpha\gamma+\beta\gamma)-8(\alpha+\beta+\gamma)+3=-97$<br>$\Rightarrow 16\left(-\frac{19}{a}\right)-8\left(-\frac{b}{a}\right)+3=-97$ | M1 | 1.1b |
| | $\Rightarrow 16\left(-\frac{19}{a}\right)-8(3)+3=-97 \Rightarrow a=...$ | M1 | 3.1a |
| | New product $64(\alpha\beta\gamma)-16(\alpha\beta+\alpha\gamma+\beta\gamma)+4(\alpha+\beta+\gamma)-1=-c$<br>$\Rightarrow 64\left(\frac{b}{a}\right)-16\left(-\frac{19}{a}\right)+4(3)-1=-c \Rightarrow c=...$ | M1 | 1.1b |
| | $a=4 \quad b=-12 \quad c=105$ | A1 | 1.1b |
| | | **(6)** | |

**Alternative Notes**

**B1:** Selects the method of giving at least two correct equations containing $\alpha$, $\beta$ and $\gamma$

**M1:** Applies the process of finding the new sum to generate an equation in *a* and *b*. Must be substituting in the correct places.

**M1:** Attempts the new pair sum to generate another equation connecting *a* and *b*. Must be substituting in the correct places.

**M1:** Solves their equations to find a value for *a* or *b*.

**M1:** Uses the new product with their values to find values for *a*, *b* and *c*

**A1:** $a=4 \quad b=-12 \quad c=105$

**1.** $\mathrm{f}(z) = z^3 + az + 52$ where $a$ is a real constant

Given that $2 - 3\mathrm{i}$ is a root of the equation $\mathrm{f}(z) = 0$

(a) write down the other complex root.

**(1)**

(b) Hence

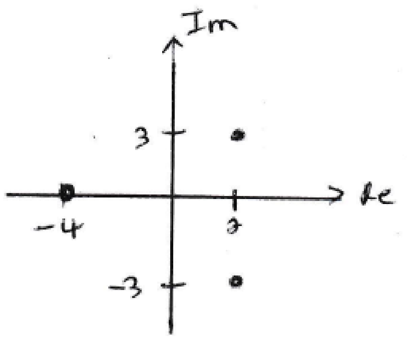
(i) solve completely $\mathrm{f}(z) = 0$

(ii) determine the value of $a$

**(4)**

(c) Show all the roots of the equation $\mathrm{f}(z) = 0$ on a single Argand diagram.

**(1)**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **1(a)** | $2+3i$ | B1 | 1.1b |
| | | **(1)** | |
| **(b) (i)** | $z*=2+3i$ so $z+z*=4$, $zz*=13$<br>$z+z*+\alpha=0 \Rightarrow \alpha=...$ or $\alpha zz*=-52 \Rightarrow \alpha=-\frac{52}{"13"}=...$ or<br>$z^2-(\text{sum roots})z+(\text{product roots})=0$ or $\left(z-(2+3i)\right)\left(z-(2-3i)\right)=...$<br>$\Rightarrow (z^2-4z+13)\underline{(z+4)} \Rightarrow z=...$ | M1 | 3.1a |
| | $z=2\pm 3\text{i},\ -4$ | A1 | 1.1b |
| **(ii)** | $(z^2-4z+13)(z+4)$ expands the brackets to find value for $a$<br>Or $a$ = pair sum = $-4(2+3i+2-3i)+13=...$<br>Or $f(-4)/f(2\pm 3i)=0 \Rightarrow ... \Rightarrow a=...$ | M1 | 1.1b |
| | $a=-3$ | A1 | 2.2a |
| | | **(4)** | |
| **(c)** |  | B1ft | 1.1b |
| | | **(1)** | |
| | | | **(6 marks)** |

| Notes: |
|---|
| **(a)**<br>**B1:** $2+3i$ |
| **(b)**<br>**(i)**<br>**M1:** A complete method to find the third root. E.g. forms the quadratic factor and uses this to find the linear factor leading to roots. Alternatively uses sum of roots = 0 or product of roots = ±52 (condone sign error) with their complex roots to find the third. Note they may have used the factor theorem to find $a$ first, which is fine. If they have found $a$ first, then the correct third root seen implies this mark. The method may be implied by the third root seen on the diagram.<br>**A1:** Correct roots, all three must be clearly stated somewhere in (b), not just seen on a diagram in part (c).<br>**(ii)**<br>**M1:** Complete method to find a value for $a$ e.g. multiplies out their quadratic and linear factors to find the coefficient of $z$, or uses pair sum, or uses factor theorem with one of the roots (may be done before finding the third root) but must reach a value for $a$.<br>**A1:** Deduces the correct value of $a$. May be seen as the $z$ coefficient in the cubic (need not be extracted, but if it is it must be correct). |
| **(c)**<br>**B1ft:** Correctly plots all three roots following through their third root in part (b). Must be labelled with the "$-4$" further from $O$ than 2, but don't be concerned about $x$ and $y$ scale. If correct look for one root on the negative real axis, with the other two symmetric about real axis in quadrants 1 and 4, but follow through their real root if positive. Accept (0,−4) labelled on the real axis in correct place as a label. |

**6.** The cubic equation

$$4x^3 + px^2 - 14x + q = 0$$

where $p$ and $q$ are real positive constants, has roots $\alpha$, $\beta$ and $\gamma$

Given that $\alpha^2 + \beta^2 + \gamma^2 = 16$

(a) show that $p = 12$

**(3)**

Given that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{14}{3}$

(b) determine the value of $q$

**(3)**

Without solving the cubic equation,

(c) determine the value of $(\alpha - 1)(\beta - 1)(\gamma - 1)$

**(4)**





P 7 1 8 0 1 A 0 1 8 3 2

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **6(a)** | $4x^3 + px^2 - 14x + q = 0 \Rightarrow x^3 + \frac{p}{4}x^2 - \frac{14}{4}x + \frac{q}{4} = 0$<br>$\alpha + \beta + \gamma = -\frac{p}{4}$ $\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{14}{4}$ or $-\frac{7}{2}$ | B1 | 3.1a |
| | $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$<br>$\left(-\frac{p}{4}\right)^2 = 16 + 2\left(-\frac{7}{2}\right) \Rightarrow p = \ldots$<br>or<br>$(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma) = \alpha^2 + \beta^2 + \gamma^2$<br>$\left(-\frac{p}{4}\right)^2 - 2\left(-\frac{7}{2}\right) = 16 \Rightarrow p = \ldots$ | M1 | 3.1a |
| | $p = 12$ * cso | A1* | 1.1b |
| | | **(3)** | |
| **(b)** | $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ | M1 | 1.1b |
| | $\frac{\left(-\frac{7}{2}\right)}{\left(\frac{-q}{4}\right)} = \frac{14}{3} \Rightarrow q = \ldots$ | M1 | 1.1b |
| | $q = 3$ | A1 | 1.1b |
| | | **(3)** | |
| | **Alternative**<br>$4\left(\frac{1}{w}\right)^3 + 12\left(\frac{1}{w}\right)^2 - 14\left(\frac{1}{w}\right) + q\{= 0\}$ | M1 | 1.1b |
| | $qw^3 - 14w^2 + 12w + 4 = 0 \Rightarrow \frac{14}{3} = -\frac{-14}{q} \Rightarrow q = \ldots$ | M1 | 1.1b |
| | $q = 3$ | A1 | 1.1b |
| | | **(3)** | |
| **(c)** | $(\alpha - 1)(\beta - 1)(\gamma - 1) = \ldots$<br>$= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) - 1$ | M1<br>A1 | 1.1a<br>1.1b |
| | $= \left(-\frac{\text{their } 3}{4}\right) - \left(-\frac{7}{2}\right) + \left(-\frac{12}{4}\right) - 1 = \ldots$ | dM1 | 1.1b |
| | $= -\frac{5}{4}$ | A1 | 1.1b |
| | | **(4)** | |
| **Alt** | $4(x + 1)^3 + 12(x + 1)^2 - 14(x + 1) + \text{'3'}\{= 0\}$ or substitutes in 1 | M1 | 1.1a |
| | $= \ldots 4 + \ldots 12 + \ldots -14 + \text{'3'} = 5$ or $4x^3 + 24x^2 + 22x + 2 +$ 'their $q$' | A1ft | 1.1b |

| | | | |
|---|---|---|---|
| | $= -\frac{\text{'their constant'}}{4}$ | dM1 | 1.1b |
| | $= -\frac{5}{4}$ | A1 | 1.1b |
| | | | |
| **(10 marks)** | | | |

**Notes:**

**(a)**
**B1:** Identifies the correct values for the sum and pair sum. This may be implied by substituting into an equation, it must be clear
**M1:** Uses the correct identity and values of their sum **and** their pair sum to find a value of $p$
**A1*:** $p = 12$ cso there is no need to see a reason

**(b)**
**M1:** Establishes a correct identity
**M1:** Uses their identity and their pair sum and their product of roots to find a value of $q$. Condone a slip but the intention must be clear.
**A1:** $q = 3$ Allow this mark from incorrect sign of both pair sum and product

**Alternative**
**M1:** Uses $x = \frac{1}{w}$ the substitution
**M1:** Simplifies to an quartic equation of the form $aw^3 + bw^2 + cw + d = 0$ and uses $\frac{14}{3} = -\frac{b}{a}$ to find a value for $q$
**A1:** $q = 3$

**(c)**
**M1:** Attempts to multiply out the three brackets.
**A1:** Correct expansion.
**dM1:** Dependent on previous method**.** Substitutes in the value of their sum, pair sum and the value of their product as appropriate. Condone a slip but the intention must be clear
**A1:** Correct value
<u>**Alternative**</u>
**M1**: Substitutes ($x$ + 1) or $x$ = 1 into the cubic with their value of $q$. Allow the use of different letters e.g. ($w$ + 1)
**A1ft:** Correct constant terms, follow through on their value of $q$
**dM1:** Applies $-\frac{\text{'their constant'}}{4}$
**A1:** Correct value

**2.** The cubic equation

$$z^3 - 3z^2 + z + 5 = 0$$

has roots $\alpha$, $\beta$ and $\gamma$.

Without solving the equation, find the cubic equation whose roots are $(2\alpha + 1)$, $(2\beta + 1)$ and $(2\gamma + 1)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where $p$, $q$ and $r$ are integers to be found.

**(5)**



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| Question | Scheme | Marks | AOs |
|---|---|---|---|
| 2 | $w=2z+1 \Rightarrow z=\frac{w-1}{2}$ | B1 | 3.1a |
| | $\left(\frac{w-1}{2}\right)^3-3\left(\frac{w-1}{2}\right)^2+\left(\frac{w-1}{2}\right)+5=0$ | M1 | 3.1a |
| | $\frac{1}{8}\left(w^3-3w^2+3w-1\right)-\frac{3}{4}\left(w^2-2w+1\right)+\frac{w-1}{2}+5=0$ | | |
| | $w^3-9w^2+19w+29=0$ | M1<br>A1<br>A1 | 1.1b<br>1.1b<br>1.1b |
| | | **(5)** | |
| **ALT 1** | $\alpha+\beta+\gamma=3, \alpha\beta+\beta\gamma+\alpha\gamma=1, \alpha\beta\gamma=-5$ | B1 | 3.1a |
| | New sum = $2\left(\alpha+\beta+\gamma\right)+3=9$<br>New pair sum = $4\left(\alpha\beta+\beta\gamma+\gamma\alpha\right)+4\left(\alpha+\beta+\gamma\right)+3=19$<br>New product = $8\alpha\beta\gamma+4\left(\alpha\beta+\beta\gamma+\gamma\alpha\right)+2\left(\alpha+\beta+\gamma\right)+1=-29$ | M1 | 3.1a |
| | $w^3-9w^2+19w+29=0$ | M1<br>A1<br>A1 | 1.1b<br>1.1b<br>1.1b |
| | | **(5)** | |
| | | **(5 marks)** | |

**Notes**

B1: Selects the method of making a connection between $z$ and $w$ by writing $z=\frac{w-1}{2}$

M1: Applies the process of substituting their $z=\frac{w-1}{2}$ into $z^3-3z^2+z+5=0$

(Allow $z = 2w + 1$)

M1: Manipulates their equation into the form $w^3+pw^2+qw+r\left(=0\right)$ having substituted their $z$ in terms of $w$. Note that the "= 0" can be missing for this mark.

A1: At least two of $p$, $q$, $r$ correct. Note that the "= 0" can be missing for this mark.

A1: Fully correct equation including "= 0"

The first 4 marks are available if another letter is used instead of $w$ but the final answer must be in terms of $w$.

**ALT1**

B1: Selects the method of giving three correct equations containing α, β and γ

M1: Applies the process of finding the new sum, new pair sum, new product

M1: Applies $w^3-\left(\text{new sum}\right)w^2+\left(\text{new pair sum}\right)w-\left(\text{new product}\right)\left(=0\right)$

**or** identifies $p$ as –(new sum) $q$ as (new pair sum) and $r$ as –(new product)

A1: At least two of $p$, $q$, $r$ correct.

A1: Fully correct equation including "= 0"

The first 4 marks are available if another letter is used instead of $w$ but the final answer must be in terms of $w$.

**7.**

$$f(z) = z^3 + z^2 + pz + q$$

where $p$ and $q$ are real constants.

The equation $f(z) = 0$ has roots $z_1$, $z_2$ and $z_3$
When plotted on an Argand diagram, the points representing $z_1$, $z_2$ and $z_3$ form the vertices of a triangle of area 35

Given that $z_1 = 3$, find the values of $p$ and $q$.

**(7)**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| 7 | Complex roots are e.g. $\alpha \pm \beta\mathrm{i}$<br>or<br>$(z^3+z^2+pz+q)\div(z-3)=z^2+4z+p+12$<br>or<br>$\mathrm{f}(3)=0 \Rightarrow 3^3+3^2+3p+q=0$<br>or<br>One of: $3+z_2+z_3=-1,\ \ 3z_2z_3=-q,\ \ 3z_2+3z_3+z_2z_3=p$ | B1 | 3.1a |
| | Sum of roots $\alpha+\beta i+\alpha-\beta i+3=-1 \Rightarrow \alpha=\ldots$<br>or<br>$\alpha+\beta i+\alpha-\beta i=-4 \Rightarrow \alpha=\ldots$ | M1 | 1.1b |
| | $\alpha=-2$ | A1 | 1.1b |
| | So $\frac{1}{2}\times 2\beta\times 5=35 \Rightarrow \beta=7$ | M1 | 1.1b |
| | $q=-3(\text{"}-2+7\mathrm{i}\text{"})(\text{"}-2-7\mathrm{i}\text{"})=\ldots$<br>or<br>$p=3(\text{"}-2+7\mathrm{i}\text{"})+3(\text{"}-2-7\mathrm{i}\text{"})+(\text{"}-2+7\mathrm{i}\text{"})(\text{"}-2-7\mathrm{i}\text{"})$<br>or<br>$(z-3)(z-(\text{"}-2+7\mathrm{i}\text{"}))(z-(\text{"}-2-7\mathrm{i}\text{"}))=\ldots$ | M1 | 3.1a |
| | $q=-159$ or $p=41$ | A1 | 1.1b |
| | $3p+q=-36 \Rightarrow p=\frac{-36-q}{3}=41$ and $q=-159$ | A1 | 1.1b |
| | | **(7)** | |
| | **Alternative** | | |
| | $(z^3+z^2+pz+q)\div(z-3)=z^2+4z+p+12$ | B1 | 3.1a |
| | $z^2+4z+p+12=0 \Rightarrow z=\frac{-4\pm\sqrt{4^2-4(p+12)}}{2}\left(=-2\pm\mathrm{i}\sqrt{p+8}\right)$ | M1 | 1.1b |
| | $\alpha=-2$ | A1 | 1.1b |
| | $\beta=\sqrt{p+8}$ | M1 | 1.1b |
| | $\frac{1}{2}\times(3+2)\times 2\sqrt{p+8}=35 \Rightarrow p=\ldots$ | M1 | 3.1a |
| | $p=41$ | A1 | 1.1b |
| | $3p+q=-36 \Rightarrow q=-159$ | A1 | 1.1b |
| | | **(7)** | |
| | | | **(7 marks)** |

| Notes |
| --- |
| B1: Recognises that the other roots must form a conjugate pair **or** obtains $z^2+4z+p+12$ (or $z^2+4z-\frac{q}{3}$) as the quadratic factor **or** writes down a correct equation for *p* and *q* **or** writes down a correct equation involving "$z_2$" and "$z_3$"<br>M1: Uses the sum of the roots of the cubic or the sum of the roots of their quadratic to find a value for "$\alpha$"<br>A1: Correct value for "$\alpha$"<br>M1: Uses their value for "$\alpha$" and the given area to find a value for "$\beta$". Must be using the area and triangle dimensions correctly e.g. $\frac{1}{2}\times\beta\times5=35\Rightarrow\beta=14$ scores M0<br>M1: Uses an appropriate method to find *p* or *q*<br>A1: A correct value for *p* or *q*<br>A1: Correct values for *p* and *q*<br>**Alternative**<br>B1: Obtains $z^2+4z+p+12$ (or $z^2+4z-\frac{q}{3}$) as the quadratic factor<br>M1: Solves their quadratic factor by completing the square or using the quadratic formula<br>A1: Correct value for "$\alpha$"<br>M1: Uses their imaginary part to find "$\beta$" in terms of *p*<br>M1: Draws together the fact that the imaginary parts of their complex conjugate pair and the real root form the sides of the required triangle and forms an equation in terms of *p*, sets equal to 35 and solves for *p*<br>A1: A correct value for *p* or *q*<br>A1: Correct values for *p* and *q* |

**2.** The cubic equation

$$2x^3 + 6x^2 - 3x + 12 = 0$$

has roots $\alpha$, $\beta$ and $\gamma$.

Without solving the equation, find the cubic equation whose roots are $(\alpha + 3)$, $(\beta + 3)$ and $(\gamma + 3)$, giving your answer in the form $pw^3 + qw^2 + rw + s = 0$, where $p$, $q$, $r$ and $s$ are integers to be found.

**(5)**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| 2. | $\{w = x+3 \Rightarrow\}\ x = w-3$ | B1 | 3.1a |
| | $2(w-3)^3 + 6(w-3)^2 - 3(w-3) + 12\ (=0)$ | M1 | 1.1b |
| | $2w^3 - 18w^2 + 54w - 54 + 6(w^2 - 6w + 9) - 3w + 9 + 12 (=0)$ | | |
| | $2w^3 - 12w^2 + 15w + 21 = 0$<br>(So $p = 2$, $q = -12$, $r = 15$ and $s = 21$) | M1<br>A1<br>A1 | 3.1a<br>1.1b<br>1.1b |
| | | **(5)** | |
| **ALT 1** | $\alpha + \beta + \gamma = -\frac{6}{2} = -3,\ \alpha\beta + \beta\gamma + \alpha\gamma = -\frac{3}{2},\ \alpha\beta\gamma = -\frac{12}{2} = -6$ | B1 | 3.1a |
| | sum roots $= \alpha + 3 + \beta + 3 + \gamma + 3$<br>$= \alpha + \beta + \gamma + 9 = -3 + 9 = 6$<br>pair sum $= (\alpha+3)(\beta+3) + (\alpha+3)(\gamma+3) + (\beta+3)(\gamma+3)$<br>$= \alpha\beta + \alpha\gamma + \beta\gamma + 6(\alpha+\beta+\gamma) + 27$<br>$= -\frac{3}{2} + 6\times -3 + 27 = \frac{15}{2}$<br>product $= (\alpha+3)(\beta+3)(\gamma+3)$<br>$= \alpha\beta\gamma + 3(\alpha\beta + \alpha\gamma + \beta\gamma) + 9(\alpha+\beta+\gamma) + 27$<br>$= -6 + 3\times -\frac{3}{2} + 9\times -3 + 27 = -\frac{21}{2}$ | M1 | 3.1a |
| | $w^3 - 6w^2 + \frac{15}{2}w - \left(-\frac{21}{2}\right)\ (=0)$ | M1 | 1.1b |
| | $2w^3 - 12w^2 + 15w + 21 = 0$<br>(So $p = 2$, $q = -12$, $r = 15$ and $s = 21$) | A1<br>A1 | 1.1b<br>1.1b |
| | | **(5)** | |
| | **(5 marks)** | | |

## Notes

| | | |
|---|---|---|
| | **B1** | Selects the method of making a connection between $x$ and $w$ by writing $x = w-3$ |
| | **M1** | Applies the process of substituting their $x = aw \pm b$ into $2x^3 + 6x^2 - 3x + 12\ (=0)$<br>So accept e.g. if $x = \frac{w}{3}$ is used. |
| | **M1** | Depends on having attempted substituting either $x = w-3$ or $x = w+3$ into the equation. This mark is for manipulating their resulting equation into the form $pw^3 + qw^2 + rw + s(=0)$ ($p \neq 0$). The "= 0" may be implied for this. |
| **See note** | **A1** | At least three of $p$, $q$, $r$ and $s$ are correct in an equation with integer coefficients. (need not have "= 0") |
| | **A1** | Correct final equation, including "=0". Accept integer multiples. |
| **ALT 1** | **B1** | Selects the method of giving three correct equations each containing $\alpha$, $\beta$ and $\gamma$. |
| | **M1** | Applies the process of finding sum roots, pair sum and product. |
| | **M1** | Applies $w^3 - (\text{their sum roots})w^2 + (\text{their pair sum})w - (\text{their product})\ (=0)$<br>Must be correct identities, but if quoted allow slips in substitution, but the "=0" may be implied. |
| **See note** | **A1** | At least three of $p$, $q$, $r$ and $s$ are correct in an equation with integer coefficients. (need not have "=0") |
| | **A1** | Correct final equation, including "=0". Accept multiples with integer coefficients. |

Note: may use another variable than $w$ for the first four marks, but the final equation must be in terms of $w$

**Notes: Do not isw the final two A marks – if subsequent division by 2 occurs then mark the final answer.**

**5.**

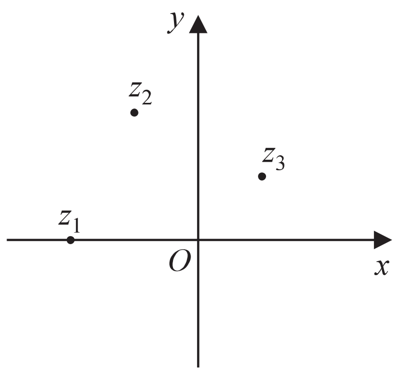



**Figure 1**

The complex numbers $z_1 = -2$, $z_2 = -1 + 2\text{i}$ and $z_3 = 1 + \text{i}$ are plotted in Figure 1, on an Argand diagram for the complex plane with $z = x + \text{i}y$

(a) Explain why $z_1$, $z_2$ and $z_3$ cannot all be roots of a quartic polynomial equation with real coefficients.

**(2)**

(b) Show that $\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \frac{\pi}{4}$

**(3)**

(c) Hence show that $\arctan(2) - \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$

**(2)**

A copy of Figure 1, labelled Diagram 1, is given on page 12.

(d) Shade, on Diagram 1, the set of points of the complex plane that satisfy the inequality

$$|z + 2| \leqslant |z - 1 - \text{i}|$$

**(2)**

**Question 5 continued**

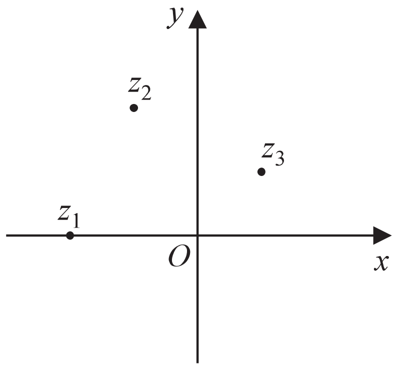
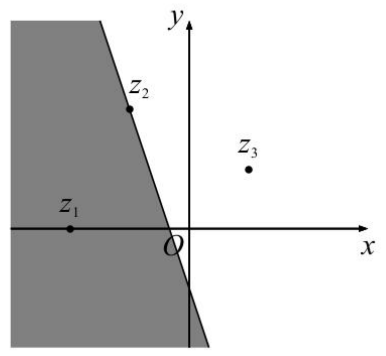



**Diagram 1**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **5 (a)** | Complex roots of a real polynomial occur in conjugate pairs | M1 | 1.2 |
| | so a polynomial with $z_1$, $z_2$ and $z_3$ as roots also needs $z_2$* and $z_3$* as roots, so 5 roots in total, but a quartic has at most 4 roots, so no quartic can have $z_1$, $z_2$ and $z_3$ as roots. | A1 | 2.4 |
| | | **(2)** | |
| **(b)** | $\frac{z_2 - z_1}{z_3 - z_1} = \frac{-1+2\mathrm{i}-(-2)}{1+\mathrm{i}-(-2)} = \frac{1+2\mathrm{i}}{3+\mathrm{i}} \times \frac{3-\mathrm{i}}{3-\mathrm{i}} = \ldots$ | M1 | 1.1b |
| | $= \frac{3-\mathrm{i}+6\mathrm{i}+2}{9+1} = \frac{5+5\mathrm{i}}{10} = \frac{1}{2}+\frac{1}{2}\mathrm{i}$ oe | A1 | 1.1b |
| | As $\frac{1}{2}+\frac{1}{2}\mathrm{i}$ is in the first quadrant (may be shown by diagram), hence $\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arctan\left(\frac{1/2}{1/2}\right)\left(= \arctan(1)\right) = \frac{\pi}{4}$ * | A1* | 2.1 |
| | | **(3)** | |
| **(c)** | $\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg\left(z_2 - z_1\right) - \arg\left(z_3 - z_1\right) = \arg\left(1+2\mathrm{i}\right) - \arg\left(3+\mathrm{i}\right)$ | M1 | 1.1b |
| | Hence $\arctan\left(2\right) - \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$ * | A1* | 2.1 |
| | | **(2)** | |
| **(d)** |   Line passing through $z_2$ and the negative imaginary axis drawn. | B1 | 1.1b |
| | Area below and left of their line shaded, where the line must have negative gradient passing through negative imaginary axis but need not pass through $z_2$ | B1 | 1.1b |
| | **Unless otherwise indicated by the student mark Diagram 1(if used) if there are multiple attempts.** | | |
| | | **(2)** | |
| | **(9 marks)** | | |

| Notes | | |
|---|---|---|
| **(a)** | **M1** | Some evidence that complex roots occur as conjugate pairs shown, e.g. stated as in scheme, or e.g. identifying if $-1+2\mathrm{i}$ is a root then so is $-1-2\mathrm{i}$. Mere mention of complex conjugates is sufficient for this mark. |
| | **A1** | A complete argument, referencing that a quartic has at most 4 roots, but would need at least 5 for all of $z_1$, $z_2$ and $z_3$ as roots.<br>There should be a clear statement about the number of roots of a quartic (e.g a quartic has four roots), and that this is not enough for the two conjugate pairs and real root. |
| **(b)** | **M1** | Substitutes the numbers in expression and attempts multiplication of numerator and denominator by the conjugate of their denominator or uses calculator to find the quotient. (May be implied.)<br>NB Applying the difference of arguments and using decimals is M0 here. |
| | **A1** | Obtains $\frac{1}{2}+\frac{1}{2}\mathrm{i}$. (May be from calculator.) Accepted equivalent Cartesian forms. |
| | **A1*** | Uses arctan on their quotient and makes reference to first quadrant or draws diagram to show they are in the first quadrant. to justify the argument. |
| **(c)** | **M1** | Applies the formula for the argument of a difference of complex numbers and substitutes values (may go directly to arctans if the arguments have already been established). If used in (b) it must be seen or referred to in (c) for this mark to be awarded. Allow for $\arg(z_2-z_1)-\arg(z_3-z_1)$ if $z_2-z_1$ and $z_3-z_1$ have been clearly identified in earlier work. |
| | **A1*** | Completes the proof clearly by identifying the required arguments and using the result of (b). Use of decimal approximations is A0. |
| **(d)** | **B1** | Draws a line through $z_2$ and passing through negative imaginary axis. |
| | **B1** | Correct side of bisector shaded. Allow this mark if the line does not pass through $z_2$. But it should be an attempt at the perpendicular bisector of the other two points – so have negative gradient and pass through the negative real axis.<br><br>Ignore any other lines drawn for these two marks. |

7. $$f(z) = z^3 - 8z^2 + pz - 24$$

where $p$ is a real constant.

Given that the equation $f(z) = 0$ has distinct roots

$$\alpha,\ \beta \text{ and } \left(\alpha + \frac{12}{\alpha} - \beta\right)$$

(a) solve completely the equation $f(z) = 0$

**(6)**

(b) Hence find the value of $p$.

**(2)**

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| Question | Scheme | Marks | AOs |
|---|---|---|---|
| 7. (a) | $\alpha+\beta+\left(\alpha+\frac{12}{\alpha}-\beta\right)=8$ so $2\alpha+\frac{12}{\alpha}=8$ | M1<br>A1 | 1.1b<br>1.1b |
| | $\Rightarrow 2\alpha^2-8\alpha+12=0$ or $\alpha^2-4\alpha+6=0$<br>$\Rightarrow \alpha=\frac{4\pm\sqrt{(-4)^2-4(1)(6)}}{2(1)}$ or $(\alpha-2)^2-4+6=0 \Rightarrow \alpha=...$ | M1 | 1.1b |
| | $\Rightarrow \alpha = 2\pm i\sqrt{2}$ are the two complex roots | A1 | 1.1b |
| | A correct full method to find the third root. Common methods are:<br>Sum of roots = 8 $\Rightarrow$ third root $=8-\left(2+i\sqrt{2}\right)-\left(2-i\sqrt{2}\right)=...$<br>third root $=2+i\sqrt{2}+\frac{12}{2+i\sqrt{2}}-\left(2-i\sqrt{2}\right)=...$<br>Product of roots = 24 $\Rightarrow$ third root $=\frac{24}{\left(2+i\sqrt{2}\right)\left(2-i\sqrt{2}\right)}=...$<br>$(z-\alpha)(z-\beta)=z^2-4z+6\Rightarrow f(z)=\left(z^2-4z+6\right)(z-\gamma)\Rightarrow\gamma=...$<br>(or long division to find third factor). | M1 | 3.1a |
| | Hence the roots of $f(z)=0$ are $2\pm i\sqrt{2}$ and 4 | A1 | 1.1b |
| | | **(6)** | |
| **(b)** | E.g. f(4) = 0 $\Rightarrow 4^3-8\times4^2+4p-24=0\Rightarrow p=...$<br>Or $p=\left(2+i\sqrt{2}\right)\left(2-i\sqrt{2}\right)+4\left(2+i\sqrt{2}\right)+4\left(2-i\sqrt{2}\right)\Rightarrow p=...$<br>Or $f(z)=(z-4)(z^2-4z+6)\Rightarrow p=...$ | M1 | 3.1a |
| | $\Rightarrow p=22$ **cso** | A1 | 1.1b |
| | | **(2)** | |
| | **(8 marks)** | | |

## Notes

| | | |
|---|---|---|
| **(a)** | **M1** | Equates sum of roots to 8 and obtains an equation in just $\alpha$. |
| | **A1** | Obtains a correct equation in $\alpha$. |
| | **M1** | Forms a three term quadratic equation in $\alpha$ and attempts to solve this equation by either completing the square or using the quadratic formula to give $\alpha=....$ |
| | **A1** | $\alpha=2\pm i\sqrt{2}$ |
| | **M1** | Any correct method for finding the remaining root. There are various routes possible. See scheme for common ones.<br>Allow this mark if –24 is used as the product.<br>See note below for a less common approach. |
| | **A1** | Third root found with all three roots correct. Note $\alpha$ and $\beta$ need not be identified. |
| **(b)** | **M1** | Any correct method of finding $p$. For example, applies the factor theorem, process of finding the pair sum of roots, or uses the roots to form f($z$). |
| | **A1** | $p=22$ by correct solution only. Note: this can be found using only their complex roots from (a) (e.g. by factor theorem) |

**Note for (a) final M –** it is possible to find the second and third roots using only one initial root (e.g. if second root forgotten or error leads to only one initial root being found).

Product of roots = $\alpha\beta\left(\alpha+\frac{12}{\alpha}-\beta\right)=24\Rightarrow\alpha\beta^2-\left(\alpha^2+12\right)\beta+24=0$, substitutes in $\alpha$ and attempts to solve the quadratic in $\beta$ to achieve remaining roots. The final M can be gained once three roots in total have been obtained. (This is unlikely to be seen as part of a correct answer.) Allow if −24 has been used for the product.

**7.**

$$f(z) = z^4 + az^3 + bz^2 + cz + d$$

where $a$, $b$, $c$ and $d$ are real constants.

The equation $f(z) = 0$ has complex roots $z_1$, $z_2$, $z_3$ and $z_4$
When plotted on an Argand diagram, the points representing $z_1$, $z_2$, $z_3$ and $z_4$ form the vertices of a square, with one vertex in each quadrant.
Given that $z_1 = 2 + 3\text{i}$, determine the values of $a$, $b$, $c$ and $d$.

**(6)**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| 7 | $z_2 = 2-3\text{i}$ | B1 | 1.1b |
| | $(z_3 =)\ p-3\text{i}$ and $(z_4 =)\ p+3\text{i}$ May be seen in an Argand diagram | M1 | 3.1a |
| | $(z_3 =) -4-3\text{i}$ and $(z_4 =) -4+3\text{i}$ May be seen in an Argand diagram, but the complex numbers used in their method takes precedence | A1 | 1.1b |
| | $(z^2-4z+13)(z^2+8z+25)$<br>or<br>$(z-(2-3\text{i}))(z-(2+3\text{i}))(z-(-4-3\text{i}))(z-(-4+3\text{i}))$<br>or<br>$a = -[(2-3\text{i})+(2+3\text{i})+(-4-3\text{i})+(-4+3\text{i})]$<br>and<br>$b = (2-3\text{i})(2+3\text{i})+(2-3\text{i})(-4-3\text{i})+(2-3\text{i})(-4+3\text{i})$ $+(2+3\text{i})(-4-3\text{i})+(2+3\text{i})(-4+3\text{i})+(-4-3\text{i})(-4+3\text{i})$<br>and<br>$c = -\left[\begin{array}{l}(2-3\text{i})(2+3\text{i})(-4-3\text{i})+(2-3\text{i})(2+3\text{i})(-4+3\text{i}) \\ +(2-3\text{i})(-4-3\text{i})(-4+3\text{i})+(2+3\text{i})(-4-3\text{i})(-4+3\text{i})\end{array}\right]$<br>and<br>$d = (2-3\text{i})(2+3\text{i})(-4-3\text{i})(-4+3\text{i})$<br>or<br>Substitutes in one root from each conjugate pair and equates real and imaginary parts and solves simultaneously<br>$(2\pm3\text{i})^4 + a(2\pm3\text{i})^3 + b(2\pm3\text{i})^2 + c(2\pm3\text{i}) + d = 0$<br>$(-4\pm3\text{i})^4 + a(-4\pm3\text{i})^3 + b(-4\pm3\text{i})^2 + c(-4\pm3\text{i}) + d = 0$ | dM1 | 3.1a |
| | $a = 4, b = 6, c = 4, d = 325$<br>$\text{f}(z) = z^4 + 4z^3 + 6z^2 + 4z + 325$ | A1<br>A1 | 1.1b<br>1.1b |
| | | **(6)** | |
| | | | **(6 marks)** |

**Notes:**

**B1:** Seen $2-3\text{i}$

**M1:** Finds the third and fourth roots of the form $p\pm3\text{i}$. May be seen in an Argand diagram

**A1:** Third and fourth roots are $-4\pm3\text{i}$. May be seen in an Argand diagram

**dM1:** Uses an appropriate method to find $\text{f}(z)$. If using roots of a polynomial at least 3 coefficients must be attempted.

**A1:** At least two of *a*, *b*, *c*, *d* correct

**A1:** All *a*, *b*, *c* and *d* correct

Note: Using roots $2 \pm 3\mathrm{i}$ and $-2 \pm 3\mathrm{i}$ leads to $z^4 + 10z^2 + 169$ Maximum score **B1 M1 A0 M1 A0 A0**

**9.** The cubic equation

$$3x^3 + x^2 - 4x + 1 = 0$$

has roots $\alpha$, $\beta$, and $\gamma$.

Without solving the cubic equation,

(a) determine the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **(3)**

(b) find a cubic equation that has roots $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$, giving your answer in the form

$x^3 + ax^2 + bx + c = 0$, where $a$, $b$ and $c$ are integers to be determined. **(3)**

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| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **9(a)** | $\alpha\beta\gamma = -\frac{1}{3}$ and $\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{4}{3}$ | B1 | 3.1a |
| | $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{-4/3}{-1/3}$ | M1 | 1.1b |
| | $= 4$ | A1 | 1.1b |
| | | **(3)** | |
| **(b)** | $\left\{\alpha + \beta + \gamma = -\frac{1}{3}\right\}$<br><br>New product $= \frac{1}{\alpha} \times \frac{1}{\beta} \times \frac{1}{\gamma} = \frac{1}{\alpha\beta\gamma} = \frac{1}{-1/3} = ...(-3)$<br><br>New pair sum $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma} = \frac{\gamma + \alpha + \beta}{\alpha\beta\gamma} = \frac{-1/3}{-1/3} = ...(1)$ | M1 | 3.1a |
| | $x^3 - (\text{part (a)})x^2 + (\text{new pair sum})x - (\text{new product})(=0)$ | M1 | 1.1b |
| | $x^3 - 4x^2 + x + 3 = 0$ | A1 | 1.1b |
| | | **(3)** | |
| | **Alternative**<br>e.g. $z = \frac{1}{x} \Rightarrow \frac{3}{x^3} + \frac{1}{x^2} - \frac{4}{x} + 1 = 0$ | M1 | 3.1a |
| | $x^3 - 4x^2 + x + 3 = 0$ | M1<br>A1 | 1.1b<br>1.1b |
| | | **(3)** | |
| | | | **(6 marks)** |

**Notes:**

**(a)**

**B1:** Correct values for the product and pair sum of the roots

**M1:** A complete method to find the sum of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. Must substitute in their values of the product and pair sum

**A1:** correct value 4

Note: If candidate does not divide by 3 so that $\alpha\beta\gamma = -1$ and $\alpha\beta + \alpha\gamma + \beta\gamma = -4$ the maximum they can score is B0 M1 A0

**(b)**

**M1:** A correct method to find the value of the new pair sum and the value of the new product

**M1:** Applies $x^3 - (\text{part (a)})x^2 + (\text{their new pair sum})x - (\text{their new product})(=0)$

**A1:** Fully correct equation, in any variable, including = 0

**(b) Alternative**

**M1:** Realises the connection between the roots and substitutes into the cubic equation

**M1:** Manipulates their equation into the form $x^3 + ax^2 + bx + c = 0$

**A1:** Fully correct equation in any variable, including = 0

**2.** The cubic equation

$$9x^3 - 5x^2 + 4x + 7 = 0$$

has roots $\alpha$, $\beta$ and $\gamma$.

Without solving the equation, find the cubic equation whose roots are $(3\alpha - 2)$, $(3\beta - 2)$ and $(3\gamma - 2)$, giving your answer in the form $aw^3 + bw^2 + cw + d = 0$, where $a$, $b$, $c$ and $d$ are integers to be determined.

**(5)**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **2** | $w=3x-2 \Rightarrow x=\dfrac{w+2}{3}$ | B1 | 3.1a |
| | $9\left(\dfrac{w+2}{3}\right)^3-5\left(\dfrac{w+2}{3}\right)^2+4\left(\dfrac{w+2}{3}\right)+7=0$ | M1 | 3.1a |
| | $\dfrac{1}{3}\left(w^3+6w^2+12w+8\right)-\dfrac{5}{9}\left(w^2+4w+4\right)+\dfrac{4}{3}\left(w+2\right)+7=0$ | | |
| | $3w^3+13w^2+28w+91=0$ | dM1<br>A1<br>A1 | 1.1b<br>1.1b<br>1.1b |
| | | **(5)** | |
| | **Alternative:** | | |
| | $\alpha+\beta+\gamma=\dfrac{5}{9}, \alpha\beta+\beta\gamma+\alpha\gamma=\dfrac{4}{9}, \alpha\beta\gamma=-\dfrac{7}{9}$ | B1 | 3.1a |
| | New sum = $3\left(\alpha+\beta+\gamma\right)-6=-\dfrac{13}{3}$ | M1 | 3.1a |
| | New pair sum = $9\left(\alpha\beta+\beta\gamma+\gamma\alpha\right)-12\left(\alpha+\beta+\gamma\right)+12=\dfrac{28}{3}$ | | |
| | New product = $27\alpha\beta\gamma-18\left(\alpha\beta+\beta\gamma+\gamma\alpha\right)+12\left(\alpha+\beta+\gamma\right)-8=-\dfrac{91}{3}$ | | |
| | $w^3-\left(-\dfrac{13}{3}\right)w^2+\dfrac{28}{3}w-\left(-\dfrac{91}{3}\right)=0$ | dM1 | 1.1b |
| | $3w^3+13w^2+28w+91=0$ | A1<br>A1 | 1.1b<br>1.1b |
| | | **(5)** | |
| | | | **(5 marks)** |

**Notes**

B1: Selects the method of making a connection between *x* and *w* by writing $x=\dfrac{w+2}{3}$

Condone the use of a different letter than *w*

M1: Applies the process of substituting $x=\dfrac{w+2}{3}$ into $9x^3-5x^2+4x+7=0$

dM1: Depends on the previous M mark. Manipulates their equation into the form $aw^3+bw^2+cw+d\left(=0\right)$. Condone the use of a different letter then *w* consistent with B1 mark.

A1: At least two of *a*, *b*, *c*, *d* correct
A1: Fully correct equation, must be in terms of *w*

**Alternative:**

B1: Selects the method of giving three correct equations containing $\alpha$, $\beta$ and $\gamma$
M1: Applies the process of finding the new sum, new pair sum, new product
dM1: Depends on the previous M mark. Applies $w^3-\left(\text{new sum}\right)w^2+\left(\text{new pair sum}\right)w-\left(\text{new product}\right)\left(=0\right)$ condone the use of any letter here.

A1: At least two of *a*, *b*, *c*, *d* correct
A1: Fully correct equation in term of *w*

**7.** $$f(z) = z^4 - 6z^3 + pz^2 + qz + r$$

where $p$, $q$ and $r$ are real constants.

The roots of the equation $f(z) = 0$ are $\alpha$, $\beta$, $\gamma$ and $\delta$ where $\alpha = 3$ and $\beta = 2 + i$

Given that $\gamma$ is a complex root of $f(z) = 0$

(a) (i) write down the root $\gamma$,

(ii) explain why $\delta$ must be real.

**(2)**

(b) Determine the value of $\delta$.

**(2)**

(c) Hence determine the values of $p$, $q$ and $r$.

**(3)**

(d) Write down the roots of the equation $f(-2z) = 0$

**(2)**

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| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **7(a)(i)** | $2-\text{i}$ | B1 | 1.2 |
| **(ii)** | Roots of polynomials with real coefficients occur in conjugate pairs. $\beta$ and $\gamma$ form a conjugate pair, $\alpha$ is real so $\delta$ must also be real.<br>or<br>Quartics have either 4 real roots, 2 real roots and 2 complex roots or 4 complex roots. As 2 complex roots and 1 real root therefore so $\delta$ must also be real. | B1 | 2.4 |
| | | **(2)** | |
| **(b)** | $\alpha+\beta+\gamma+\delta=6$<br>$\Rightarrow 3+2+\text{i}+2-\text{i}+\delta=6 \Rightarrow \delta=\ldots$ | M1 | 3.1a |
| | $\delta=-1$ | A1 | 1.1b |
| | | **(2)** | |
| **(c)** | $\text{f}(z)=(z-3)(z+1)(z-(2+\text{i}))(z-(2-\text{i}))=\ldots$<br>Alternative<br>pair sum $=(3)(2+\text{i})+(3)(2-\text{i})+(3)(-1)+(-1)(2+\text{i})$<br>$+(-1)(2-\text{i})+(2+\text{i})(2-\text{i})=\ldots\{10\}$<br>triple sum $=(3)(2+\text{i})(2-\text{i})+(3)(-1)(2+\text{i})$<br>$+(3)(-1)(2-\text{i})+(-1)(2+\text{i})(2-\text{i})=\ldots\{-2\}$<br>product $=(3)(2+\text{i})(2-\text{i})(-1)=\ldots\{-15\}$ | M1 | 3.1a |
| | $=(z^2-2z-3)(z^2-4z+5)$<br>$=z^4-6z^3+10z^2+2z-15$ | A1<br>A1 | 1.1b<br>1.1b |
| | | **(3)** | |
| **(d)** | $z=\frac{1}{2},-\frac{3}{2}$ | B1ft | 1.1b |
| | $z=-1\pm\frac{\text{i}}{2}$ | B1ft | 1.1b |
| | | **(2)** | |
| | | | **(9 marks)** |

**Notes**

(a)(i)
B1: Correct complex number
(a)(ii)
B1: Correct explanation
(b)
M1: Uses 2 ± i and 1 together with the sum of roots = ± 6 to find a value for $\delta$
A1: Correct value
(c)
M1: Uses ($z$ – 3) and ($z$ – their $\delta$) and their conjugate pair correctly as factors and makes an attempt to expand
Alternatively attempts to find the pair sum, triple sum and product
A1: Establishes at least 2 of the required coefficients correctly
A1: Correct quartic or correct constants
(d)

B1ft: For $-\frac{3}{2}$ and $-\frac{\delta}{2}$ as the real roots

B1ft: For $-1-\frac{i}{2}$ and $-\frac{\gamma}{2}$ as the complex roots

4. The roots of the quartic equation

$$3x^4 + 5x^3 - 7x + 6 = 0$$

are $\alpha, \beta, \gamma$ and $\delta$

Making your method clear and without solving the equation, determine the exact value of

(i) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

**(3)**

(ii) $\dfrac{2}{\alpha} + \dfrac{2}{\beta} + \dfrac{2}{\gamma} + \dfrac{2}{\delta}$

**(3)**

(iii) $(3 - \alpha)(3 - \beta)(3 - \gamma)(3 - \delta)$

**(3)**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| **4(i)** | $\sum \alpha_i = -\frac{5}{3}$ and $\sum \alpha_i \alpha_j = 0$<br>This mark can be awarded if seen in part (ii) or part (iii) | B1 | 3.1a |
| | So $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2\left(\sum \alpha_i \alpha_j\right) = ...$ | M1 | 1.1b |
| | $= \frac{25}{9} - 2 \times 0 = \frac{25}{9}$ | A1 | 1.1b |
| | | **(3)** | |
| **(ii)** | $\sum \alpha_i \alpha_j \alpha_k = \frac{7}{3}$ and $\prod \alpha_i = 2$ or for $x = \frac{2}{w}$ used in equation<br>This mark can be awarded if seen in part (i) or part (iii) | B1 | 2.2a |
| | So $2\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right) = 2 \times \frac{\sum \alpha_i \alpha_j \alpha_k}{\alpha\beta\gamma\delta} = 2 \times \frac{'\frac{7}{3}'}{'\frac{6}{3}'}$ or for<br>$3\left(\frac{16}{w^4}\right) + 5\left(\frac{8}{w^3}\right) - 7\left(\frac{2}{w}\right) + 6 = 0 \Rightarrow 6w^4 - 14w^3 + ... = 0$ leading to $\frac{14}{6}$ | M1 | 1.1b |
| | $\left(= 2 \times \frac{7/3}{2}\right)\left(= \frac{14}{6}\right) = \frac{7}{3}$ | A1 | 1.1b |
| | | **(3)** | |
| **(iii)** | $(3-\alpha)(3-\beta)(3-\gamma)(3-\delta) = ...$ expands all four brackets<br>Or equation with these roots is $3(3-x)^4 + 5(3-x)^3 - 7(3-x) + 6 = 0$ | M1 | 3.1a |
| | $= 81 - 27\left(\sum \alpha_i\right) + 9\left(\sum \alpha_i \alpha_j\right) - 3\left(\sum \alpha_i \alpha_j \alpha_k\right) + \prod \alpha_i$<br>$= 81 - 27\left(-\frac{5}{3}\right) + 9(0) - 3\left(\frac{7}{3}\right) + 2$<br>Or expands to fourth power and constant terms and attempts product of roots $3x^4 + ... + 3 \times 3^4 + 5 \times 3^3 - 7 \times 3 + 6 \to \prod \alpha_i = \frac{"363"}{3}$ | dM1 | 1.1b |
| | $= 121$ | A1 | 1.1b |
| | | **(3)** | |
| | | | **(9 marks)** |

**Notes:**

**(i)**

**B1:** Correct sum and pair sum of roots seen or implied. Must realise the pair sum is zero.
**Note:** These values can be seen anywhere in the candidate's solution

**M1**: Uses correct expression for the sum of squares.

**A1**: $\frac{25}{9}$. Allow this mark from incorrect sign on sum of squares (but they will score B0 if the sign is incorrect).

**(ii)**

**B1:** Correct triple sum and product of roots seen or implied. May be stated in (i). Alternatively, this may be scored for sight of $x = \frac{2}{w}$ used as a transformation in the equation.

**Note:** These values can be seen anywhere in the candidate's solution

**M1:** Substitutes their values into $2 \times \frac{\sum \alpha_i \alpha_j \alpha_k}{\alpha\beta\gamma\delta} = ...$ In the alternative it is for rearranging the equation to a quartic in $w$ and uses to find the sum of the roots.

**A1:** $\frac{7}{3}$ Allow this mark from incorrect sign of both triple sum and product (but they will score B0 if the sign is incorrect).

**(iii)**

**M1:** A correct method to find the value used – may recognise structure as scheme, may expand the expression in stages, or may attempt to use a linear transformation $(3 - x)$ or e.g. $(3 - w)$ in original equation. Condone slips as long as the intention is clear.

**dM1:** Dependent on previous method mark**.** Uses at least 2 values of their sum of roots etc. in their expression. If using a linear shift this is for expanding to find the coefficient of $x^4$ and constant term and attempts product of roots by dividing the constant term by the coefficient of $x^4$.

**A1:** 121.