Cp1Ch3 XMQs and MS

(Total: 56 marks)

1. CP2_2021 Q4 . 9 marks - CP1ch3 Series
2. CP(AS)_2018 Q6 . 10 marks - CP1ch3 Series
3. CP(AS)_2019 Q6 . 9 marks - CP1ch3 Series
4. CP(AS)_2020 Q5 . 7 marks - CP1ch3 Series
5. CP(AS)_2021 Q3 . 9 marks - CP1ch3 Series
6. CP(AS)_2022 Q5 . 12 marks - CP1ch3 Series

4. In this question you may assume the results for

$$\sum_{r=1}^{n} r^3 \ , \quad \sum_{r=1}^{n} r^2 \quad \text{and} \quad \sum_{r=1}^{n} r$$

(a) Show that the sum of the cubes of the first n positive odd numbers is

$$n^2(2n^2 - 1)$$

(5)

The sum of the cubes of 10 consecutive positive odd numbers is 99 800

(b) Use the answer to part (a) to determine the smallest of these 10 consecutive positive odd numbers.

(4)

Question	Scheme	Marks	AOs
4(a)	A complete attempt to find the sum of the cubes of the first *n* odd numbers using three of the standard summation formulae. Attempts to find $\sum(2r+1)^3$ or $\sum(2r-1)^3$ by expanding and using summation formulae	M1	3.1a
	$\sum_{r=1}^{n}(2r-1)^3 = \sum_{r=1}^{n}(8r^3-12r^2+6r-1) = 8\sum_{r=1}^{n}r^3 - 12\sum_{r=1}^{n}r^2 + 6\sum_{r=1}^{n}r - \sum_{r=1}^{n}1$ or $\sum_{r=0}^{n-1}(2r+1)^3 = \sum_{r=0}^{n-1}(8r^3+12r^2+6r+1) = 8\sum_{r=0}^{n-1}r^3 + 12\sum_{r=0}^{n-1}r^2 + 6\sum_{r=0}^{n-1}r + \sum_{r=0}^{n-1}1$	M1	1.1b
	$= 8\frac{n^2}{4}(n+1)^2 - 12\frac{n}{6}(n+1)(2n+1) + 6\frac{n}{2}(n+1) - n$ or $= 8\frac{(n-1)^2}{4}(n)^2 + 12\frac{(n-1)}{6}(n)(2n-1) + 6\frac{(n-1)}{2}(n) + n$	M1 A1	1.1b 1.1b
	Multiplies out to achieve a correct intermediate line for example $n\ \ n+1\ \ 2n^2-2n+1\ \ -n = 2n^4-2n^3+n^2+2n^3-2n^2+n-n$ $2n^4+4n^3+2n^2-4n^3-6n^2-2n+3n^2+3n-n$ leading to $= n^2(2n^2-1)$ cso *	A1 *	2.1
		(5)	
(b)	$\sum_{r=n}^{n+9}(2r-1)^3 = \sum_{r=1}^{n+9}(2r-1)^3 - \sum_{r=1}^{n-1}(2r-1)^3$ $= (n+9)^2\left(2(n+9)^2-1\right) - (n-1)^2\left(2(n-1)^2-1\right) = 99800$ or $\sum_{r=n+1}^{n+10}(2r-1)^3 = \sum_{r=1}^{n+10}(2r-1)^3 - \sum_{r=1}^{n}(2r-1)^3$ $= (n+10)^2\left(2(n+10)^2-1\right) - (n)^2\left(2n^2-1\right) = 99800$ or $\sum_{r=n-9}^{n}(2r-1)^3 = \sum_{r=1}^{n}(2r-1)^3 - \sum_{r=1}^{n-10}(2r-1)^3$ $= (n)^2\left(2(n)^2-1\right) - (n-10)^2\left(2(n-10)^2-1\right) = 99800$	M1	3.1a
	$80n^3+960n^2+5820n-86760=0$ or $80n^3+1200n^2+7980n-79900=0$ or $80n^3-1200n^2+7980n-119700=0$	A1	1.1b
	Solves cubic equation	dM1	1.1b

	Achieves $n = 6$ and the smallest number as 11 or Achieves $n = 5$ and the smallest number as 11 or Achieves $n = 15$ and the smallest number as 11	A1	2.3
		(4)	
(9 marks)			

Notes:

(a)

M1: A complete attempt to find the sum of the cubes of *n* odd numbers using three of the standard summation formulae.

M1: Expands $\sum_{r=1}^{n}(2r-1)^3$ or $\sum_{r=0}^{n-1}(2r+1)^3$ and splits into fours appropriate sums.

M1: Applies the result for at least three summations $\sum_{r=0}^{n-1} r^3, \sum_{r=0}^{n-1} r^2, \sum_{r=0}^{n-1} r$ and $\sum_{r=0}^{n-1} 1$ or $\sum_{r=1}^{n} r^3, \sum_{r=1}^{n} r^2, \sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} 1$ as appropriate to their expansion provided that there is an attempt at cubing some values.

A1: Correct unsimplified expression.

A1 *: Multiplies out to achieve a correct intermediate expression which clearly leads to the correct expression. cso

Special case: If uses $\sum_{r=1}^{n}(2r+1)^3$ leading to $=8\frac{n^2}{4}(n+1)^2+12\frac{n}{6}(n+1)(2n+1)+6\frac{n}{2}(n+1)+n$ max score is M1 M0 M1 A1 A0

(b)

M1: Uses the answer to part (a) to find the **s**um of the cubes of the first *N* + 10 odd numbers minus the sum of the first *N* odd numbers and sets equal to 99800 or equivalent.

A1: Correct simplified cubic equation.

dM1: Uses their calculator to solve their cubic equation, dependent on previous method mark.

A1: cao

6. (a) Use the standard results for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that

$$\sum_{r=1}^{n}(3r-2)^2 = \frac{1}{2}n\left[6n^2 - 3n - 1\right]$$

for all positive integers n.

(5)

(b) Hence find any values of n for which

$$\sum_{r=5}^{n}(3r-2)^2 + 103\sum_{r=1}^{28} r\cos\left(\frac{r\pi}{2}\right) = 3n^3$$

(5)

Question	Scheme	Marks	AOs
6(a)	$(3r-2)^2 = 9r^2 - 12r + 4$	B1	1.1b
	$\sum_{r=1}^{n}(9r^2-12r+4) = 9\times\frac{1}{6}n(n+1)(2n+1) - 12\times\frac{1}{2}n(n+1)+...$	M1	2.1
	$= 9\times\frac{1}{6}n(n+1)(2n+1) - 12\times\frac{1}{2}n(n+1)+4n$	A1	1.1b
	$= \frac{1}{2}n\left[3(n+1)(2n+1)-12(n+1)+8\right]$	dM1	1.1b
	$= \frac{1}{2}n\left[6n^2-3n-1\right]$*	A1*	1.1b
		(5)	
(b)	$\sum_{r=5}^{n}(3r-2)^2 = \frac{1}{2}n(6n^2-3n-1) - \frac{1}{2}(4)\left(6(4)^2-3\times4-1\right)$	M1	3.1a
	$\sum_{r=1}^{28} r\cos\left(\frac{r\pi}{2}\right) = 0-2+0+4+0-6+0+8+0-10+0+12+...$	M1	3.1a
	$3n^3 - \frac{3}{2}n^2 - \frac{1}{2}n - 166 + 103\times14 = 3n^3$ $\Rightarrow 3n^2 + n - 2552 = 0$	A1	1.1b
	$\Rightarrow 3n^2+n-2552=0 \Rightarrow n = ...$	M1	1.1b
	$n = 29$	A1	2.3
		(5)	
		(10 marks)	

Notes

(a) Do not allow <u>proof by induction</u> (but the B1 could score for $(3r-2)^2 = 9r^2 - 12r + 4$ if seen)

B1: Correct expansion

M1: **<u>Substitutes</u>** at least one of the standard formulae into their expanded expression

A1: Fully correct expression

dM1: Attempts to factorise $\frac{1}{2}n$ having used at least one standard formula correctly. Dependent on the first M mark and dependent on there being an n in all terms.

A1*: Obtains the printed result with no errors seen

(b)

M1: Uses the result from part (a) by substituting n = 4 and subtracts from the result in (a) in order to find the first sum in terms of n.

M1: Identifies the periodic nature of the second sum by calculating terms. This may be implied by a sum of 14.

A1: Uses their sum and the given result to form the correct 3 term quadratic

M1: Solves their three term quadratic to obtain at least one value for n

A1: Obtains n = 29 only or obtains n = 29 and $n = -\frac{88}{3}$ and rejects the $-\frac{88}{3}$

6. An art display consists of an arrangement of n marbles.

When arranged in ascending order of mass, the mass of the first marble is 10 grams. The mass of each subsequent marble is 3 grams more than the mass of the previous one, so that the rth marble has mass $(7 + 3r)$ grams.

(a) Show that the mean mass, in grams, of the marbles in the display is given by

$$\frac{1}{2}(3n+17)$$

(3)

Given that there are 85 marbles in the display,

(b) use the standard summation formulae to find the standard deviation of the mass of the marbles in the display, giving your answer, in grams, to one decimal place.

(6)

Question	Scheme	Marks	AOs
6. (a)	(mean $=\bar{x}=$) $\frac{1}{n}\sum_{r=1}^{n}(7+3r)$	M1	1.1a
	$\sum_{r=1}^{n}(7+3r)=\left(7\sum_{r=1}^{n}1+3\sum_{r=1}^{n}r=\right)7n+3\frac{n}{2}(n+1)$	M1	1.1b
	$\bar{x}=7+\frac{3}{2}(n+1)=\frac{14+3n+3}{2}=\frac{1}{2}(3n+17)$ *	A1*	2.1
		(3)	
(b)	Correct overall strategy to find the variance or standard deviation. This must include: • An attempt to find the mean • An attempt at $\sum(7+3r)^2$ as part of their formula (however poor, or if stated and followed by a value or if used with incorrect limits). • An attempt at either variance formula with their mean (allow slips in the formula)	M1	3.1a
(Mean)	mean ($=\bar{x}$) $=136$	B1	1.1b
(Sum)	Way1: $\sum_{r=1}^{n}(7+3r)^2=\sum_{r=1}^{n}(49+42r+9r^2)$ $=\underline{\underline{49n}}+42\times\underline{\frac{1}{2}n(n+1)}+9\times\underline{\frac{1}{6}n(n+1)(2n+1)}$ Way 2: $\sum_{r=1}^{n}\left(x_i-\bar{x}\right)^2=\sum_{r=1}^{n}\left(7+3r-\text{"136"}\right)^2=a\sum_{r=1}^{n}r^2+b\sum_{r=1}^{n}r+c\sum_{r=1}^{n}1$ $=9\times\underline{\frac{1}{6}n(n+1)(2n+1)}-\text{"774"}\times\underline{\frac{1}{2}n(n+1)}+\underline{\underline{\text{"16641"}n}}$	<u>M1</u> B1	1.1b 1.1b
(Variance/standard deviation)	Way 1: $=\frac{\text{"2032690"}}{85}-136^2=...$ or $\frac{\text{"2032690"}}{84}-\frac{85}{84}\times136^2=...$ Way 2: $=\frac{\text{"460530"}}{85}=...$ or $\frac{\text{"460530"}}{84}=...$ (using sample standard deviation).	M1	1.1b
	So s.d $=\sqrt{5418}=73.6$ (g) Accept 74.0 (g) if sample s.d. used	A1	1.1b
		(6)	
	(9 marks)		

Notes		
(a)	**M1**	Selects the correct procedure for finding the mean ($\bar{x}$), attempting sum and dividing by *n*.
	M1	Splits the sum and applies the formulae for $\sum r$ (accept $7+3\frac{n}{2}(n+1)$ here) Or uses arithmetic series formula $\frac{1}{2}n(a+l)$ with *a* = 10 and *l* an attempt at 7 + 3×*n*, or $\frac{n}{2}(2a+(n-1)d)$ with *a* = 10 and *d* = 3..
	A1*	Correct work proceeding to the answer with an intermediate step shown. **Special case:** Award M0M1A0 for candidates who use $\frac{1}{2}(a+l)$ or equivalent without justification of the division by *n*.
(b)	**M1**	Correct overall strategy to get as far as the variance of marbles in the collection. The attempt at variance should be recognisable (though allow e.g sign slips in the formula for this mark) and an attempt, however poor, at $\sum (7+3r)^2$ must have been made
	B1	Correct value for the mean for 85 marbles (accept as a single fraction, $\frac{272}{2}$). If a student works algebraically until the last step, a correct final answer will imply this mark.
	M1	Expands brackets and applies summation formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to their expression, either in terms of *n* or with *n* = 85 **but must have correct limits**. Allow for obtaining an expression of the correct form for Way 2 if the mean is kept in terms of "*n*". This mark is for correct application of these two summation formula on an attempt at $\sum_{r=1}^{n} (7+3r)^2$ so accept even if this is not part of an attempt at the variance.
	B1	Correct use of $\sum_{r=1}^{n} 1 = n$ in their expression (must be correct limits).
	M1	**Correctly applies** variance or standard deviation formula with *n* = 85, their attempt at $\sum x^2$ (which need not be using 7 + 3*r* or correct limits) and their mean. Accept use of the sample variance/standard deviation is used (dividing by $n-1$) For reference the variance formula is $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) - \bar{x}^2$ where $x_r = 7+3r$ here, or accept for sample variance $\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \left(\frac{1}{n-1}\sum_{i=1}^{n} x_i^2\right) - \frac{n\bar{x}^2}{n-1}$
	A1	Correct standard deviation to 1 decimal place. If sample standard deviation is used, the answer will be 74.0 g to 1 d.p. (74.04…)
	Note: Question specifies use of summation formula and so these must be seen for the 2nd M and 2nd B mark. However, if just 2032690 appears from a calculator all other marks are available.	

5.

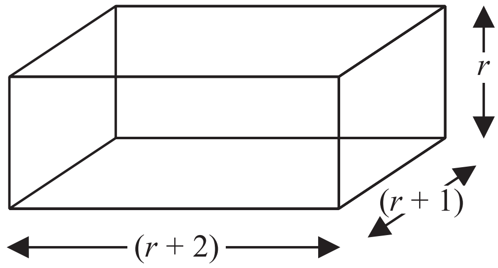

Figure 2

A block has length $(r + 2)$ cm, width $(r + 1)$ cm and height r cm, as shown in Figure 2.

In a set of n such blocks, the first block has a height of 1 cm, the second block has a height of 2 cm, the third block has a height of 3 cm and so on.

(a) Use the standard results for $\sum_{r=1}^{n} r^3, \sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that the **total** volume, V, of all n blocks in the set is given by

$$V = \frac{n}{4}(n+1)(n+2)(n+3) \qquad n \geqslant 1$$

(5)

Given that the total volume of all n blocks is

$$(n^4 + 6n^3 - 11\,710) \text{ cm}^3$$

(b) determine how many blocks make up the set.

(2)

Question	Scheme	Marks	AOs
5(a)	Volume = $r\times(r+1)\times(r+2)$	B1	1.1b
	A complete method for finding the total volume of *n* blocks and expressing it in sigma notation. This can be implied by later work. $\sum_{r=1}^{n}(r^3+3r^2+2r)$	M1	3.1b
	$V=\frac{1}{4}n^2(n+1)^2+3\times\frac{1}{6}n(n+1)(2n+1)+2\times\frac{n}{2}(n+1)$	M1	2.1
	$V=\frac{1}{4}n(n+1)\left[n(n+1)+2(2n+1)+4\right]$	dM1	1.1b
	$V=\frac{1}{4}n(n+1)\left[n^2+5n+6\right]$ $\Rightarrow V=\frac{1}{4}n(n+1)(n+2)(n+3)$ *	A1*	1.1b
		(5)	
(b)	Sets $\frac{1}{4}n(n+1)(n+2)(n+3)=n^4+6n^3-11710$ $\frac{1}{4}n^4+\frac{3}{2}n^3+\frac{11}{4}n^2+\frac{3}{2}n=n^4+6n^3-11710$ simplifies $(3n^4+18n^3-11n^2-6n-46840=0)$ and solves *n* = …	M1	1.1b
	There are 10 blocks or *n* = 10	A1	3.2a
		(2)	
			(7 marks)

Notes:

(a)
B1: Correct volume of a block
M1: Expressing the total volume of all *n* blocks as a series in terms of r, r^2 and r^3
M1: Substitutes at least one of the standard formulae into their volume.
dM1: Attempts to factorise $\frac{1}{4}n(n+1)$ having used at least one standard formula correctly. Each term must contain a factor of $n(n+1)$
A1*: Obtains the printed result with no errors seen, no bracketing errors and following from $V=\frac{1}{4}n(n+1)\left[n^2+5n+6\right]$ o.e.

Note: Going from $\frac{1}{4}n(n^3+6n^2+11n+6)$ to $\frac{1}{4}n(n+1)(n+2)(n+3)$ with no reasoning shown scores **dM0 A0**

(b)

M1: Sets the printed answer $= n^4 + 6n^3 - 11710$, simplifies, collects terms and uses their calculator to solve a quartic equation to find a value for n.
A1: Selects n = 10 or states that there are **10 blocks** from a **correct equation**

3. (a) Use the standard results for summations to show that for all positive integers n

$$\sum_{r=1}^{n}(5r-2)^2 = \frac{1}{6}n(an^2+bn+c)$$

where a, b and c are integers to be determined.

(5)

(b) Hence determine the value of k for which

$$\sum_{r=1}^{k}(5r-2)^2 = 94k^2$$

(4)

Question	Scheme	Marks	AOs
3(a)	$(5r-2)^2 = 25r^2 - 20r + 4$	B1	1.1b
	$\sum_{r=1}^{n} 25r^2 - 20r + 4 = \frac{25}{6}n(n+1)(2n+1) - \frac{20}{2}n(n+1) + ...$	M1	2.1
	$= \frac{25}{6}n(n+1)(2n+1) - \frac{20}{2}n(n+1) + 4n$	A1	1.1b
	$= \frac{1}{6}n\left[25(2n^2+3n+1) - 60(n+1) + 24\right]$	dM1	1.1b
	$= \frac{1}{6}n\left[50n^2+15n-11\right]$	A1	1.1b
		(5)	
(b)	$\frac{1}{6}k\left[50k^2+15k-11\right] = 94k^2$	M1	1.1b
	$50k^3 - 549k^2 - 11k = 0$ or $50k^2 - 549k - 11 = 0$	A1	1.1b
	$(k-11)(50k+1) = 0 \Rightarrow k = ...$	M1	1.1b
	k = 11(only)	A1	2.3
		(4)	
			(9 marks)

Notes

(a)
B1: Correct expansion
M1: Substitutes at least one of the standard formulae into their expanded expression
A1: Fully correct expression
dM1: Attempts to factorise $\frac{1}{6}n$ having used at least one standard formula correctly. Dependent on the first M mark.
A1: Obtains the correct expression or the correct values of *a*, *b* and *c*
(b)
M1: Uses their result from part (a) and sets equal to $94k^2$ and attempt to expand and collect terms.
A1: Correct cubic or quadratic
M1: Attempts to solve their 3TQ or cubic equation
A1: Identifies the correct value of *k* with no other values offered

5. (a) Use the standard summation formulae to show that, for $n \in \mathbb{N}$,

$$\sum_{r=1}^{n}\left(3r^2 - 17r - 25\right) = n\left(n^2 - An - B\right)$$

where A and B are integers to be determined.

(4)

(b) Explain why, for $k \in \mathbb{N}$,

$$\sum_{r=1}^{3k} r\tan\left(60r\right)^\circ = -k\sqrt{3}$$

(2)

Using the results from part (a) and part (b) and showing all your working,

(c) determine any value of n that satisfies

$$\sum_{r=5}^{n}\left(3r^2 - 17r - 25\right) = 15\left[\sum_{r=6}^{3n} r\tan\left(60r\right)^\circ\right]^2$$

(6)

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Question	Scheme	Marks	AOs
5(a)	$\sum_{r=1}^{n}\left(3r^2-17r-25\right)=3\times\frac{n}{6}(n+1)(2n+1)-17\times\frac{1}{2}n(n+1)-...$	M1	1.1b
	$=3\times\frac{n}{6}(n+1)(2n+1)-17\times\frac{1}{2}n(n+1)-25n$	A1	1.1b
	$=n\left(\frac{1}{2}\left(2n^2+3n+1\right)-\frac{17}{2}(n+1)-25\right)$ or $=\frac{n}{2}\left(\left(2n^2+3n+1\right)-17(n+1)-50\right)$	M1	1.1b
	$=n\left(n^2-7n-33\right)$ cso (so $A=7$ and $B=33$)	A1 cso	2.1
		(4)	
(b)	$\sum_{r=1}^{3k}r\tan\left(60r\right)°$ $=\tan\left(60\right)°+2\tan\left(120\right)°+3\tan\left(180\right)°+4\tan\left(240\right)°+5\tan\left(300\right)°$ $+6\tan\left(360\right)°+$ $=\left(\sqrt{3}-2\sqrt{3}+0\right)+\left(4\sqrt{3}-5\sqrt{3}+0\right)+...$	M1	3.1a
	Since tan has period 180° we see $\tan\left(60r\right)°$ repeats every three terms and **each group of three terms results in** $-\sqrt{3}$ **as a sum**, so with ***k*** **groups** of terms the sum is $-k\sqrt{3}$	A1	2.4
		(2)	
(c)	$\sum_{r=5}^{n}\left(3r^2-17r-25\right)=\sum_{r=1}^{n}\left(3r^2-17r-25\right)-\sum_{r=1}^{4}\left(3r^2-17r-25\right)$	M1	1.1b
	$=n\left(n^2-7n-33\right)-4\left(4^2-7\times4-33\right)$ $\left(=n\left(n^2-7n-33\right)+180\right)$	A1	1.1b
	$\sum_{r=6}^{3n}r\tan\left(60r\right)°=-n\sqrt{3}+2\sqrt{3}$ (allow for $-n\sqrt{3}--2\sqrt{3}$)	B1	2.2a
	$\Rightarrow n\left(n^2-7n-33\right)+180=15\left[-n\sqrt{3}+2\sqrt{3}\right]^2$ $\Rightarrow n^3-7n^2-33n+180=15\left(3n^2-12n+12\right)$ $\Rightarrow n^3-52n^2+147n=0$	M1	3.1a
	$\Rightarrow n^3-52n^2+147n=0\Rightarrow n=...$	M1	1.1b
	But need $n>5$ for sums to be valid, so $n=49$ (allow if $n=0$ also given but $n=3$ must be rejected).	A1	2.3
		(6)	
		(12 marks)	

Notes:

(a)

M1: Applies the formulas for sum of integers and sum of squares of integers to the summation.

A1: Correct unsimplified expression for the sum, including the $25n$

M1: Expands and factors out the n or ½n

A1: Correct proof, no errors seen.

(b)

M1: Writes out first few terms of the sum, at least 3, and identifies the repeating pattern, e.g. through bracketed terms or stating sum repeat every three terms oe.

A1: Correct explanation identifying $-\sqrt{3}$ is the sum of each group of three terms, so with k lots of three terms the sum is $-k\sqrt{3}$

(c)

M1: Applies formula from (a) to left-hand side as a difference of two summations with either 4 or 5 as the limit on the second sum.

A1: Correct expression for the left-hand side in terms of n

B1: Correct expression for the sum on the right-hand side, allow if it arises from lower limit 6 used instead of 5 as the 6^{th} term is zero. May subtract the first few terms directly from the work in (b).

M1: Both sides expanded and terms gathered to reach a simplified cubic equation for n with no other unknowns (may not have factor of n if errors made, which is fine for the method mark). This mark is not dependent on any previous marks and can be awarded as long as there is an attempt at both sides of the equation and an attempt at squaring their $\sum_{r=6}^{3n} r\tan(60r)°$.

If divides through by n this mark is awarded for a 3TQ

M1: Solves their cubic equation, which may be via calculator (so may need to check values). They may divide by n and solve a quadratic. Condone decimal roots truncated or rounded

A1: Selects the correct value of n to give 49 as the only non-trivial answer. The value 3 must be rejected as summation on left undefined for this value, but accept if 0 and 49 are given (since both sides evaluate to 0 for $n = 0$ depending on one's interpretation of summations).